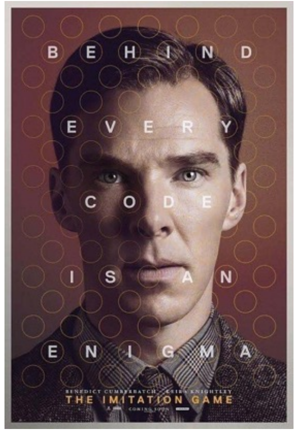
Automata & languages

A primer on the Theory of Computation



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October, 1 2015

Last week, we learned about
closure and **equivalence** of regular languages

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The class of regular languages is closed under the

- union
- concatenation
- star

regular operations

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regular operations

if L_1 and L_2 are regular, then so are

$L_1 \cup L_2$

$L_1 . L_2$

L_1^*

Last week, we learned about
closure and equivalence of regular languages

is equivalent to

$\text{DFA} \asymp \text{NFA}$

We started to look at REX,
the third way of representing regular languages

$\text{DFA} \asymp \text{NFA}$

REX

Are REX, NFA and DFA all equivalent?

$\text{DFA} \asymp \text{NFA}$

)(?

REX

We stopped asking ourselves whether all languages are regular

L_1 $\{0^n1^n \mid n \geq 0\}$

L_2 $\{w \mid w$ has an equal number of 0s and 1s$\}$

L_3 $\{w \mid w$ has an equal number of occurrences of 01 and 10$\}$

(only one of them actually is)

Advanced Automata

Thu Oct 1

1 Equivalence (the end)

- DFA
- NFA
- Regular Expression

2 Non-regular languages

3 Context-free languages

Three tough languages

1) $L_1 = \{0^n1^n \mid n \geq 0\}$
2) $L_2 = \{w \mid w$ has an equal number of 0s and 1s$\}$
3) $L_3 = \{w \mid w$ has an equal number of occurrences of 01 and 10 as substrings$\}$

Three tough languages

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- In order to fully understand regular languages, we also must understand their limitations!

Pigeonhole principle

- Consider language L, which contains word w ∈ L.
- Consider an FA which accepts L, with n < |w| states.
- Then, when accepting w, the FA must visit at least one state twice.

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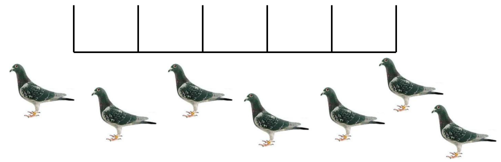
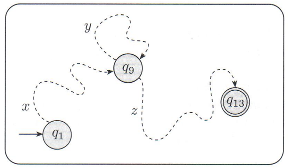
- Consider an FA which accepts L, with n < |w| states.
- Then, when accepting w, the FA must visit at least one state twice.

- This is according to the pigeonhole (a.k.a. Dirichlet) principle:
  - If m>n pigeons are put into n pigeonholes, there's a hole with more than one pigeon.
  - That's a pretty fancy name for a boring observation...



Languages with unbounded strings

- Consequently, regular languages with unbounded strings can only be recognized by FA (finite! bounded!) automata if these long strings loop.



- The FA can enter the loop once, twice, ..., and not at all.
- That is, language L contains all {xz, xyz, xy^2z, xy^3z, ...}.

Pumping Lemma

- Theorem:

  Given a regular language *L*, there is a number *p* (the pumping number)
  such that:
  any string *u* in *L* of length ≥ *p* is pumpable within its first *p* letters.

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- A string $u \in L$ with $|u| \geq p$ is pumpable if it can be split in 3 parts xyz s.t.:
  - $|y| \geq 1$ (mid-portion y is non-empty)
  - $|xy| \leq p$ (pumping occurs in first p letters)
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- If there is no such p, then the language is not regular

Pumping Lemma Example

- Let L be the language $\{0^n1^n \mid n \geq 0\}$

- Assume (for the sake of contradiction) that L is regular
- Let p be the pumping length. Let u be the string 0^p1^p.
- Let's check string u against the pumping lemma:

- "In other words, for all $u \in L$ with $|u| \geq p$ we can write:
  - $u = xyz$ (x is a prefix, z is a suffix)
  - $|y| \geq 1$ (mid-portion y is non-empty)
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Let's make the example a bit harder…

- Let L be the language {w | w has an equal number of 0s and 1s}

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Now you try…

- Is $L_1 = \{ww \mid w \in (0 \cup 1)^*\}$ regular?
- Is $L_2 = \{1^n \mid n \text{ being a prime number }\}$ regular?

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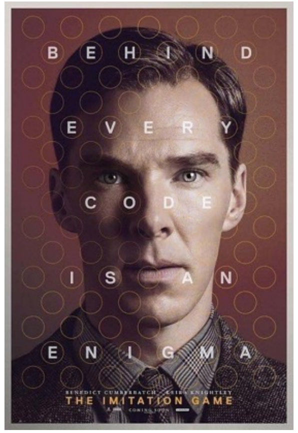

Part 1 regular language

Part 2 context-free language

Part 3 turing machine

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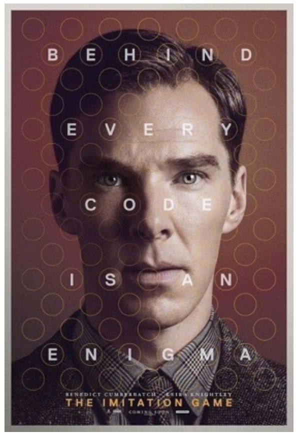

regular language

Part 2 context-free language

turing machine

Motivation

- Why is a language such as $\{0^n1^n \mid n \geq 0\}$ not regular?!?
- It's really simple! All you need to keep track is the number of 0's…
- In this chapter we first study context-free grammars
  - More powerful than regular languages
  - Recursive structure
  - Developed for human languages
  - Important for engineers (parsers, protocols, etc.)

Example

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  - In fact, all palindromes can be generated from ε using these rules.

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- Q: How would you generate 11011011?

Context Free Grammars (CFG): Definition

- Definition: A context free grammar consists of (V, Σ, R, S) with:
  - V: a finite set of variables (or symbols, or non-terminals)
  - Σ: a finite set set of terminals (or the alphabet)
  - R: a finite set of rules (or productions)
    of the form $v \rightarrow w$ with $v \in V$, and $w \in (\Sigma_\varepsilon \cup V)^*$
    (read: "v yields w" or "v produces w")
  - $S \in V$: the start symbol.

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- Q: What are (V, Σ, R, S) for our palindrome example?

Derivations and Language

- Definition: The derivation symbol "$\Rightarrow$" (read "1-step derives" or "1-step produces") is a relation between strings in $(\Sigma \cup V)^*$.
  We write $x \Rightarrow y$ if x and y can be broken up as $x = svt$ and $y = swt$ with $v \rightarrow w$ being a production in R.

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Derivations and Language

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- Definition: Let G be a context-free grammar. The context-free language (CFL) generated by G is the set of all terminal strings which are derivable from the start symbol. Symbolically: $L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$

Example: Infix Expressions

- Infix expressions involving {+, ×, a, b, c, (,)}
- E stands for an expression (most general)
- F stands for factor (a multiplicative part)
- T stands for term (a product of factors)
- V stands for a variable: a, b, or c
- Grammar is given by:
  - $E \rightarrow T \mid E + T$
  - $T \rightarrow F \mid T \times F$
  - $F \rightarrow V \mid (E)$
  - $V \rightarrow a \mid b \mid c$
- Convention: Start variable is the first one in grammar (E)

Example: Infix Expressions

- Consider the string u given by $a \times b + (c + (a + c))$
- This is a valid infix expression. Can be generated from E.

1. A sum of two expressions, so first production must be $E \Rightarrow E + T$
2. Sub-expression $a \times b$ is a product, so a term so generated by sequence $E + T \Rightarrow T + T \Rightarrow T \times F + T \Rightarrow^* a \times b + T$
3. Second sub-expression is a factor only because a parenthesized sum. $a \times b + T \Rightarrow a \times b + F \Rightarrow a \times b + (E) \Rightarrow a \times b + (E + T)$ …
4. $E \Rightarrow E + T \Rightarrow T + T \Rightarrow T \times F + T \Rightarrow F \times F + T \Rightarrow V \times F + T \Rightarrow a \times F + T \Rightarrow a \times V + T \Rightarrow a \times b + T \Rightarrow a \times b + F \Rightarrow a \times b + (E) \Rightarrow a \times b + (E + T) \Rightarrow a \times b + (T + T) \Rightarrow a \times b + (F + T) \Rightarrow a \times b + (V + T) \Rightarrow a \times b + (c + T) \Rightarrow a \times b + (c + F) \Rightarrow a \times b + (c + (E)) \Rightarrow a \times b + (c + (E + T)) \Rightarrow a \times b + (c + (T + T)) \Rightarrow a \times b + (c + (F + T)) \Rightarrow a \times b + (c + (a + T)) \Rightarrow a \times b + (c + (a + F)) \Rightarrow a \times b + (c + (a + V)) \Rightarrow a \times b + (c + (a + c))$

Left- and Right-most derivation

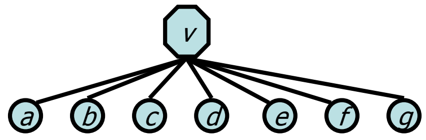
- The derivation on the previous slide was a so-called left-most derivation.
- In a right-most derivation, the variable most to the right is replaced.
  - $E \Rightarrow E + T \Rightarrow E + F \Rightarrow E + (E) \Rightarrow E + (E + T) \Rightarrow$ etc.

Ambiguity

- There can be a lot of ambiguity involved in how a string is derived.
- Another way to describe a derivation in a unique way is using derivation trees.

Derivation Trees

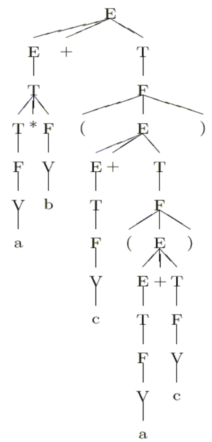
- In a derivation tree (or parse tree) each node is a symbol. Each parent is a variable whose children spell out the production from left to right. For, example *v* → *abcdefg*:



- The root is the start variable.
- The leaves spell out the derived string from left to right.

Derivation Trees

- On the right, we see a derivation tree for our string *a*×*b* + (*c* + (*a* + *c*))
- Derivation trees help understanding semantics! You can tell how expression should be evaluated from the tree.



Ambiguity

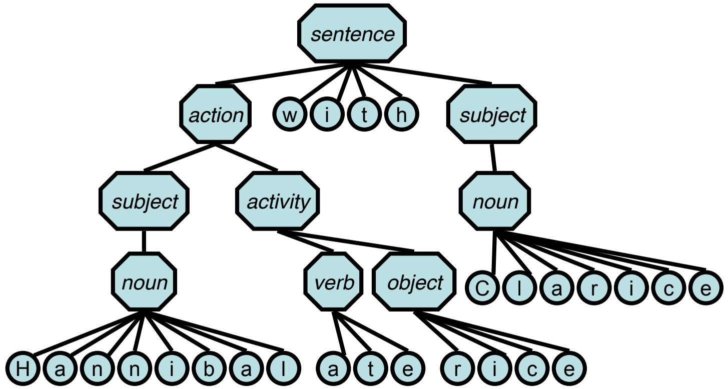
<sentence>	→	***<action>* | *<action>*** with ***<subject>***
<action>	→	***<subject><activity>***
<subject>	→	***<noun>* | *<noun>*** and ***<subject>***
<activity>	→	***<verb>* | *<verb><object>***
<noun>	→	Hannibal | Clarice | rice | onions
<verb>	→	ate | played
<prep>	→	with | and | or
<object>	→	***<noun>* | *<noun><prep><object>***

- Clarice played with Hannibal
- Clarice ate rice with onions
- Hannibal ate rice with Clarice
- Q: Are there any suspect sentences?

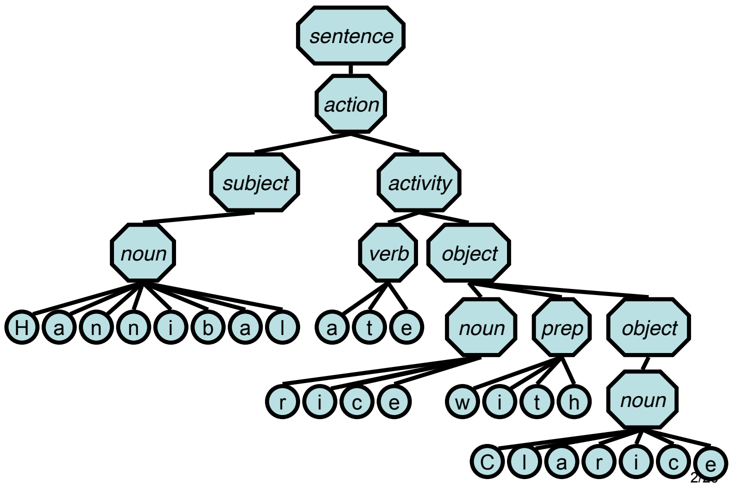
Ambiguity

- A: Consider "Hannibal ate rice with Clarice"
- This could either mean
  - Hannibal and Clarice ate rice *together*.
  - Hannibal ate rice and *ate* Clarice.
- This ambiguity arises from the fact that the sentence has two different parse-trees, and therefore two different interpretations:

Hannibal and Clarice Ate



Hannibal the Cannibal



Ambiguity: Definition

- Definition:

  A string *x* is said to be ambiguous relative the grammar *G* if there are two essentially different ways to derive *x* in *G*.
  - *x* admits two (or more) different parse-trees
  - equivalently, *x* admits different left-most [resp. right-most] derivations.
- A grammar *G* is said to be ambiguous if there is some string *x* in *L*(*G*) which is ambiguous.

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  - x admits two (or more) different parse-trees
  - equivalently, x admits different left-most [resp. right-most] derivations.

- A grammar G is said to be ambiguous if there is some string x in $L(G)$ which is ambiguous.

- Question: Is the grammar $S \rightarrow ab \mid ba \mid aSb \mid bSa \mid SS$ ambiguous?
  - What language is generated?

CFG's: Proving Correctness

- The recursive nature of CFG's means that they are especially amenable to correctness proofs.

- For example let's consider the grammar

$$G = (S \rightarrow \varepsilon \mid ab \mid ba \mid aSb \mid bSa \mid SS)$$

- We claim that L(G) = L = $\{ x \in \{a,b\}^* \mid n_a(x) = n_b(x) \}$, where $n_a(x)$ is the number of a's in x, and $n_b(x)$ is the number of b's.

- *Proof*: To prove that L = L(G) is to show both inclusions:
  i. $L \subseteq L(G)$: Every string in L can be generated by G.
  ii. $L \supseteq L(G)$: G only generate strings of L.
     - This part is easy, so we concentrate on part i.

Proving $L \subseteq L(G)$

- $L \subseteq L(G)$: Show that every string x with the same number of a's as b's is generated by G. Prove by induction on the length $n = |x|$.
- Base case: The empty string is derived by $S \rightarrow \varepsilon$.
- Inductive hypothesis: Assume $n > 0$. Let u be the smallest non-empty prefix of x which is also in L.
  - Either there is such a prefix with $|u| < |x|$, then $x = uv$ whereas $v \in L$ as well, and we can use $S \rightarrow SS$ and repeat the argument.
  - Or $x = u$. In this case notice that u can't start and end in the same letter. If it started and ended with a then write $x = ava$. This means that v *must have* 2 more b's than a's. So somewhere in v the b's of x catch up to the a's which means that there's a smaller prefix in L, contradicting the definition of u as the *smallest* prefix in L. Thus for some string v in L we have $x = avb$ OR $x = bva$. We can use either $S \rightarrow aSb$ OR $S \rightarrow bSa$.

Designing Context-Free Grammars

- As for regular languages this is a creative process.

- However, if the grammar is the union of simpler grammars, you can design the simpler grammars (with starting symbols S_1, S_2, respectively) first, and then add a new starting symbol/production $S \rightarrow S_1 \mid S_2$.

- If the CFG happens to be regular as well, you can first design the FA, introduce a variable/production for each state of the FA, and then add a rule $x \rightarrow ay$ to the CFG if $\delta(x,a) = y$ is in the FA. If a state x is accepting in FA then add $x \rightarrow \varepsilon$ to CFG. The start symbol of the CFG is of course equivalent to the start state in the FA.

- There are quite a few other tricks. Try yourself…