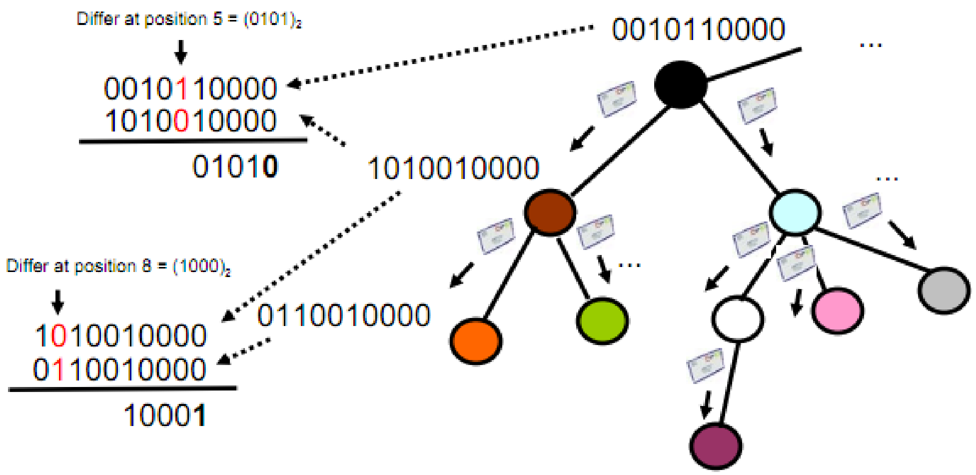
Distributed Computing over
Communication Networks:

Locality Lower Bounds

Vertex Coloring: Results so far?

E.g., on trees in log*(n) time, down to 6 colors...



Idea:

root should have label 0 (fixed)

in each step: send ID to c_v to all children;

receive c_p from parent and interpret as little-endian bit string: $c_p=c(k)...c(0)$

let i be smallest index where c_v and c_p differ

set new c_v = i (as bit string) || $c_v(i)$

until $c_v \in \{0,1,2,...,5\}$ (at most 6 colors)

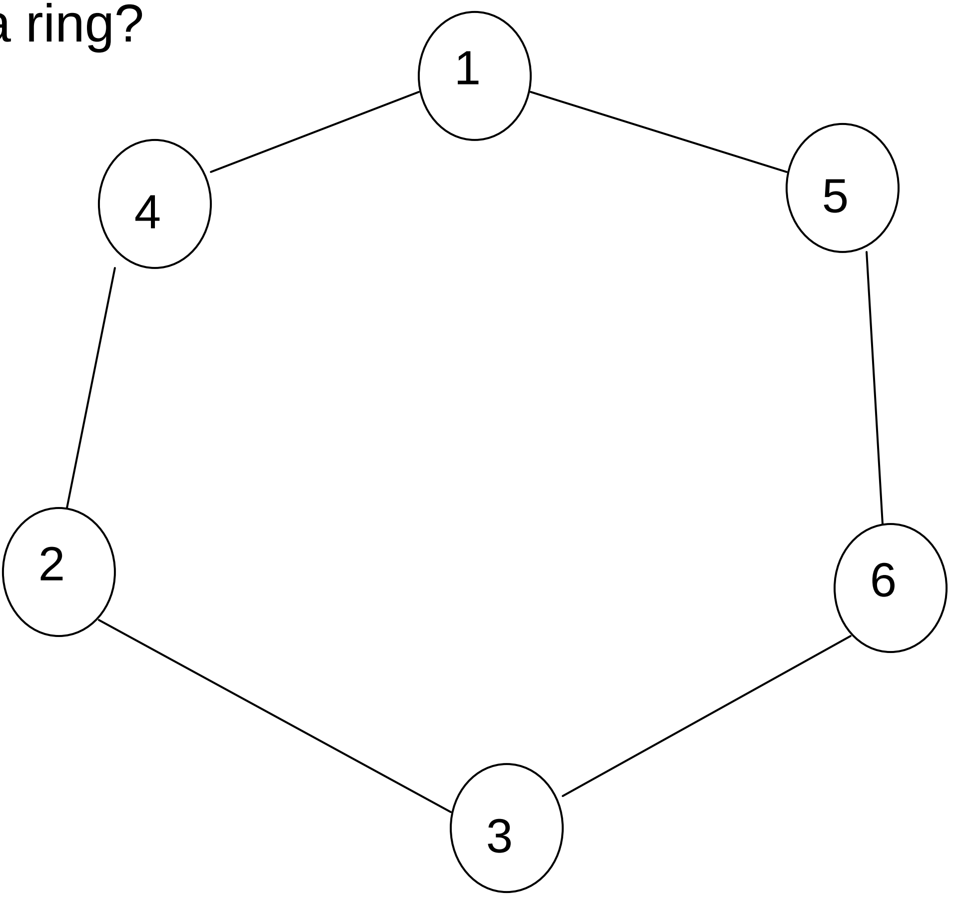
Round 1

Is this optimal??

... and then shift-down: down to 3 colors (same complexity).

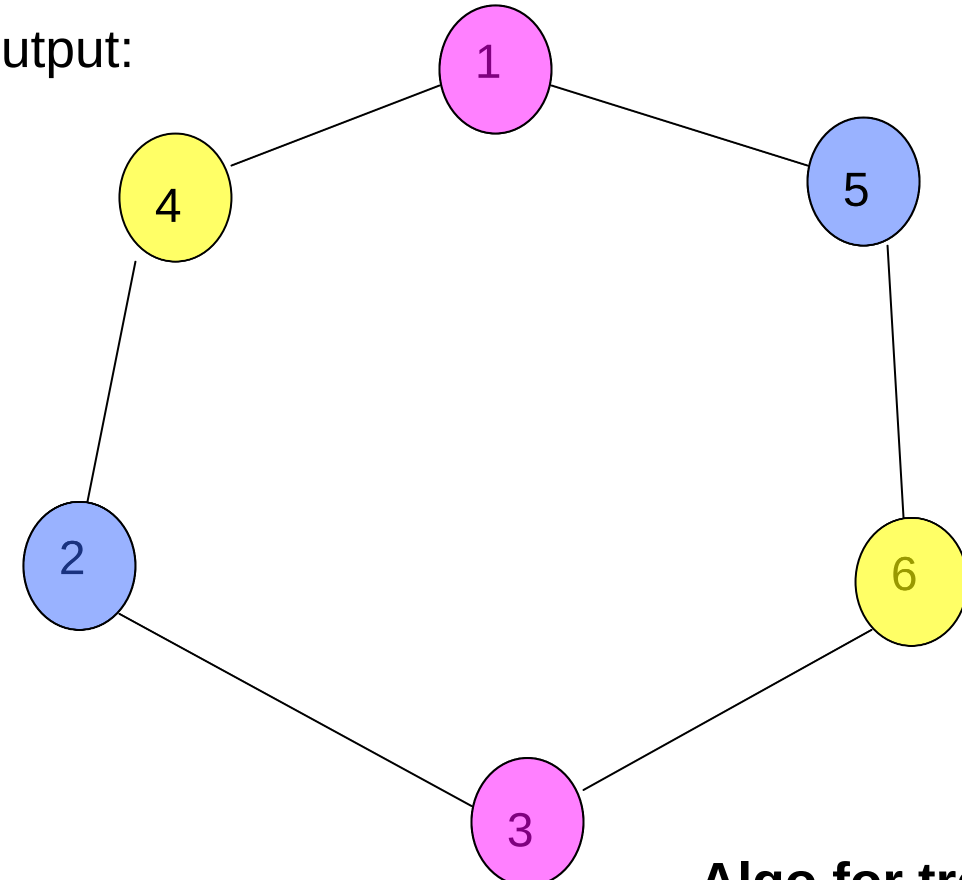
From trees to rings...

How to color a ring?



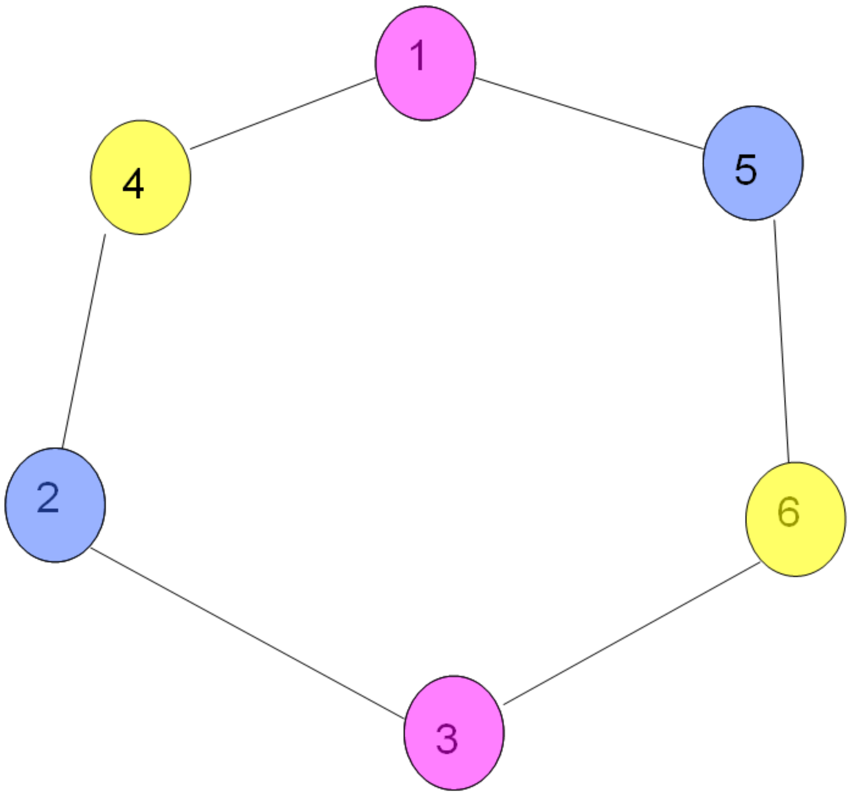
Ring Coloring

Possible output:



Algo for trees can be adapted!
So log*(n) time...!

Lower Bounds: First Thoughts and Outlook



Assume unique node IDs:

Lower bound for # colors without communication?

n

Lower bound for # colors with one communication round?

log n

Lower bound for # colors with two communication rounds?

log log n

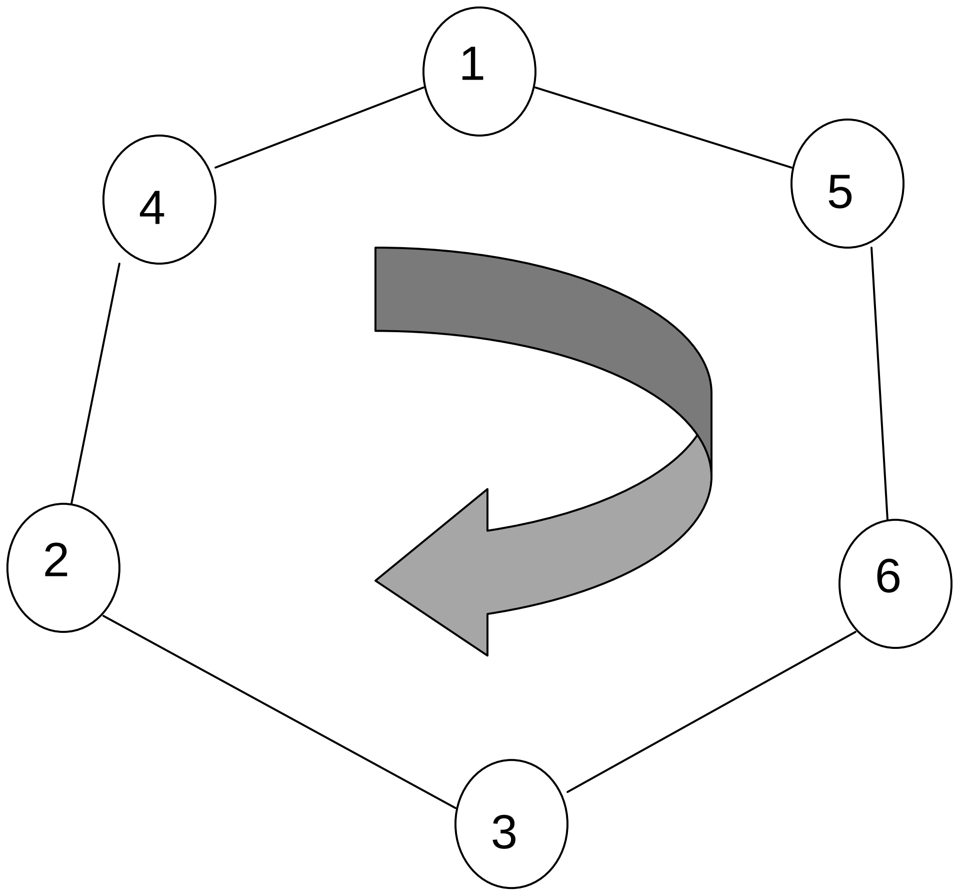
Lower bound for # colors with log* n communication rounds?

O(1)

Lower Bounds: For simple ring, not tree....

Time complexity to 3-color a ring?

Upper bound also log*(n)... But lower bround? How to prove?



Class of algos?
Need assumptions!

1. synchronous, **directed** ring
   (communication in both directions and nodes can differentiate between clockwise and counter-clockwise)
2. IDs from **1...n**
   (not in order, otherwise trivial!)
3. unbounded message size

Canonical Form of Distributed Algorithm?

What can a distributed algorithm do or learn in r rounds?

1. Initially, all nodes only know their own ID
2. As information needs at least r rounds to travel r hops, a node can only learn about r-hop neighborhood

Note that any local r-round algorithm can be brought into canonical form!

Canonical Form

1. First, in r rounds: send initial state to nodes at distance r
2. Then: compute output based on complete information about r-hop neighborhood

In other words: we can emulate any local algorithm by making all communication first and then do all local computations! Why?

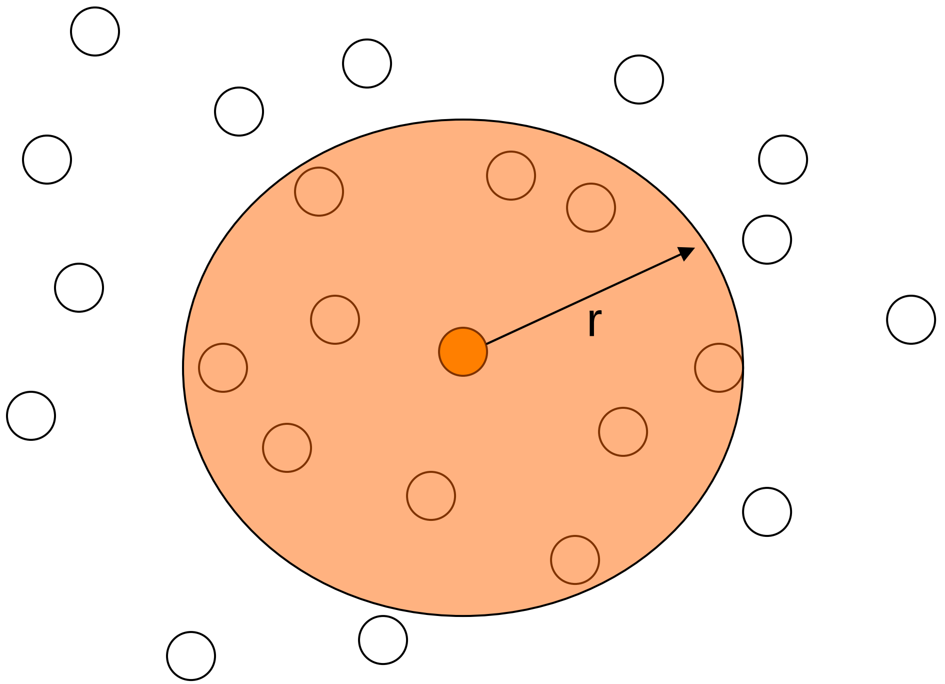
No new information can be generated by local compuations: rest can only be processing / filtering / selection...

No Deterministic Local Algorithms Can Do More...

We can do all communication first and then do all local computations!

How to prove this?

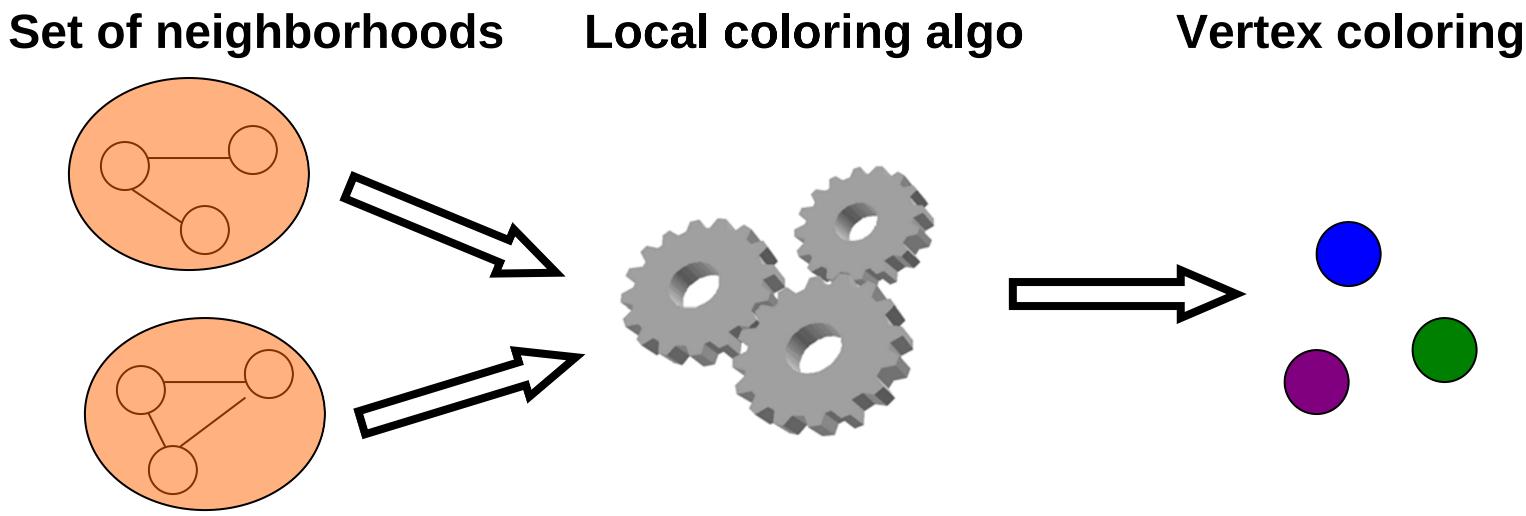
Let A be any r-round algorithm.
We can show that the canonical form algorithm C can compute all possible messages that A may send as well.
By induction over distance of nodes...:
if we can compute messages of first i rounds in (r-i+1)-neighborhood, we have all information to compute first (i+1) round messages in (r-i)-neighborhood. See „Skript“. ☺



Implication?

Takeaway

A local coloring algorithm can be seen as a function which takes neighborhoods and outputs colors.



Local Views

This motivates the following definition:

r-Hop View

We call the collection of the initial states of all nodes in the r-neighborhood of a node v the „r-hop view of v“.

Due to our canonical form lemma, this means that:

Deterministic r-Round Algo

A deterministic r-round algorithm A is a function that maps every possible r-hop view to the set of possible outputs.

Implication for nodes with same view?
Must produce same output, in any algorithm!

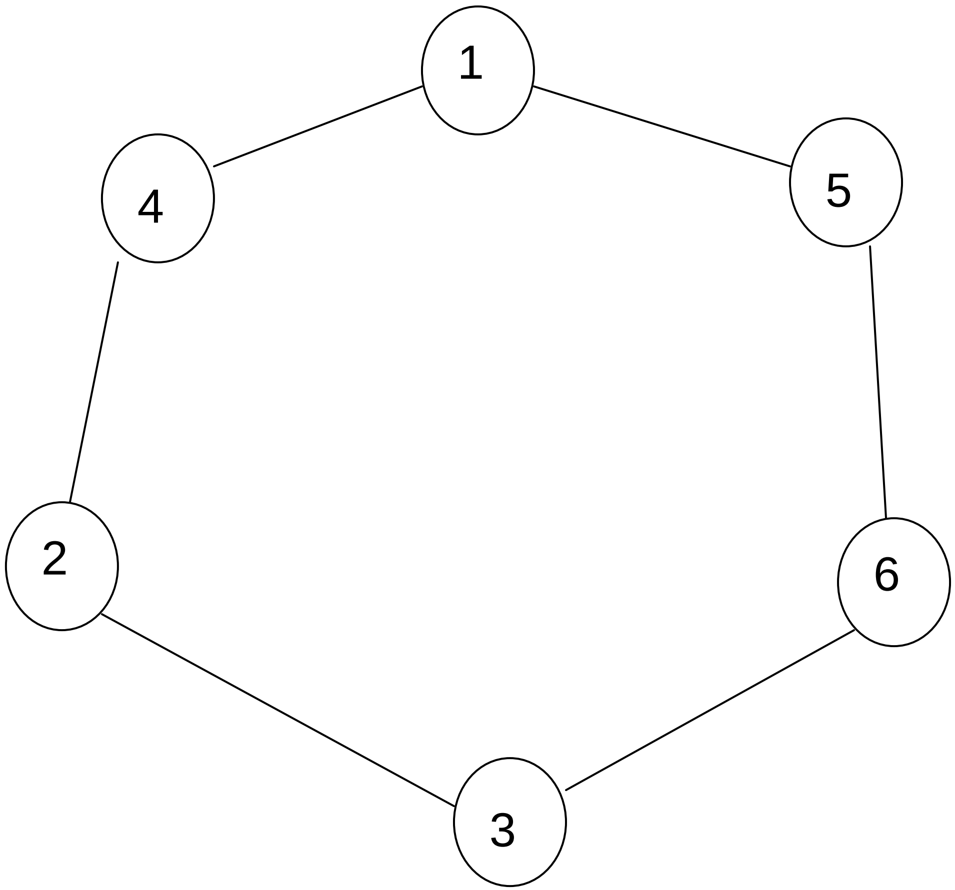
Roadmap

So if any local algorithm can be emulated
by a canonic algorithm, the question remains:

How good can a canonic algorithm maximally be?

Rings

How do r-hop views of our rings look like?
E.g., 1-hop view of 4?



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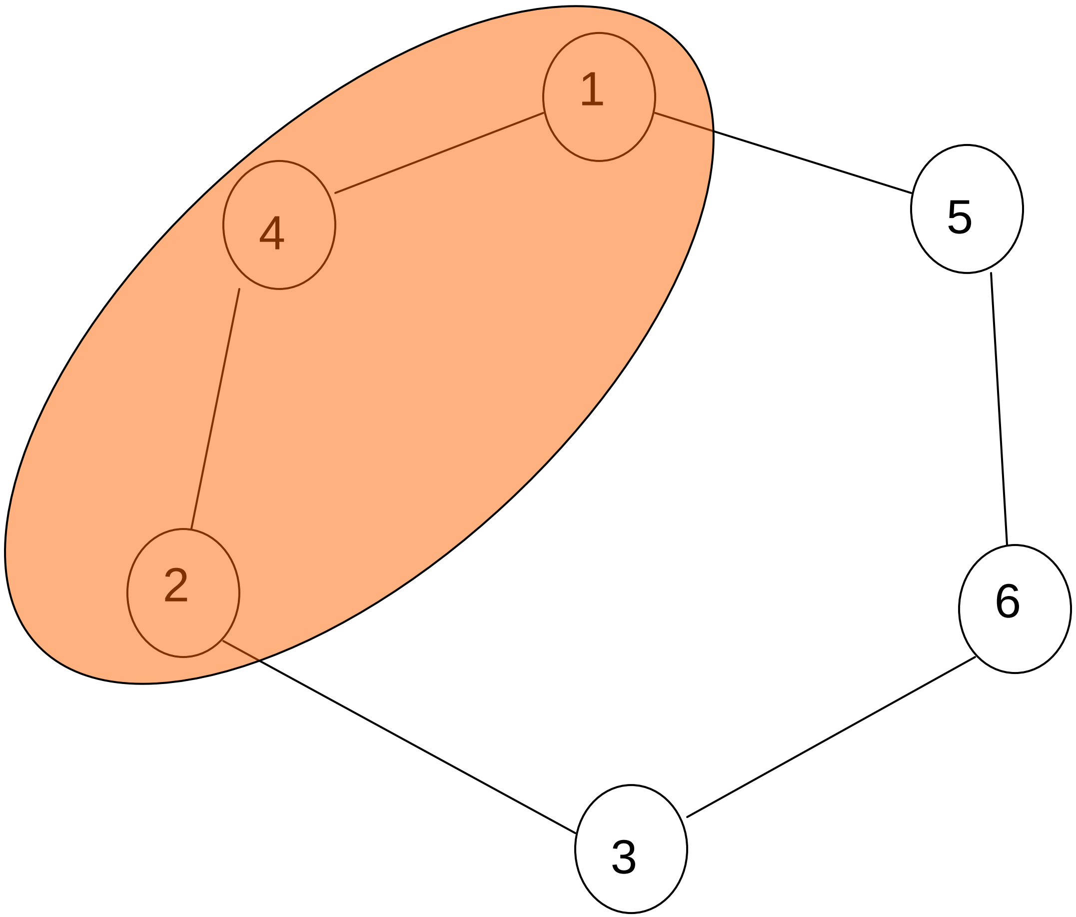
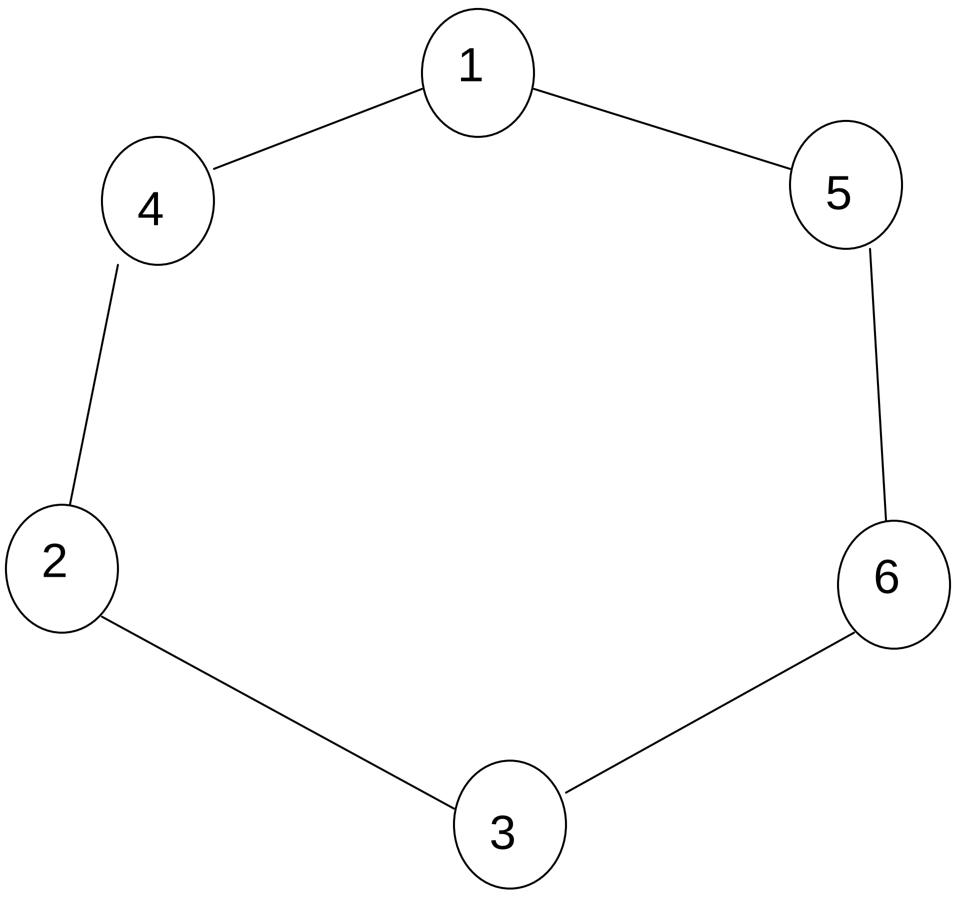


Rings

How do r-hop views of our rings look like?
E.g., 2-hop view of 4?



Rings

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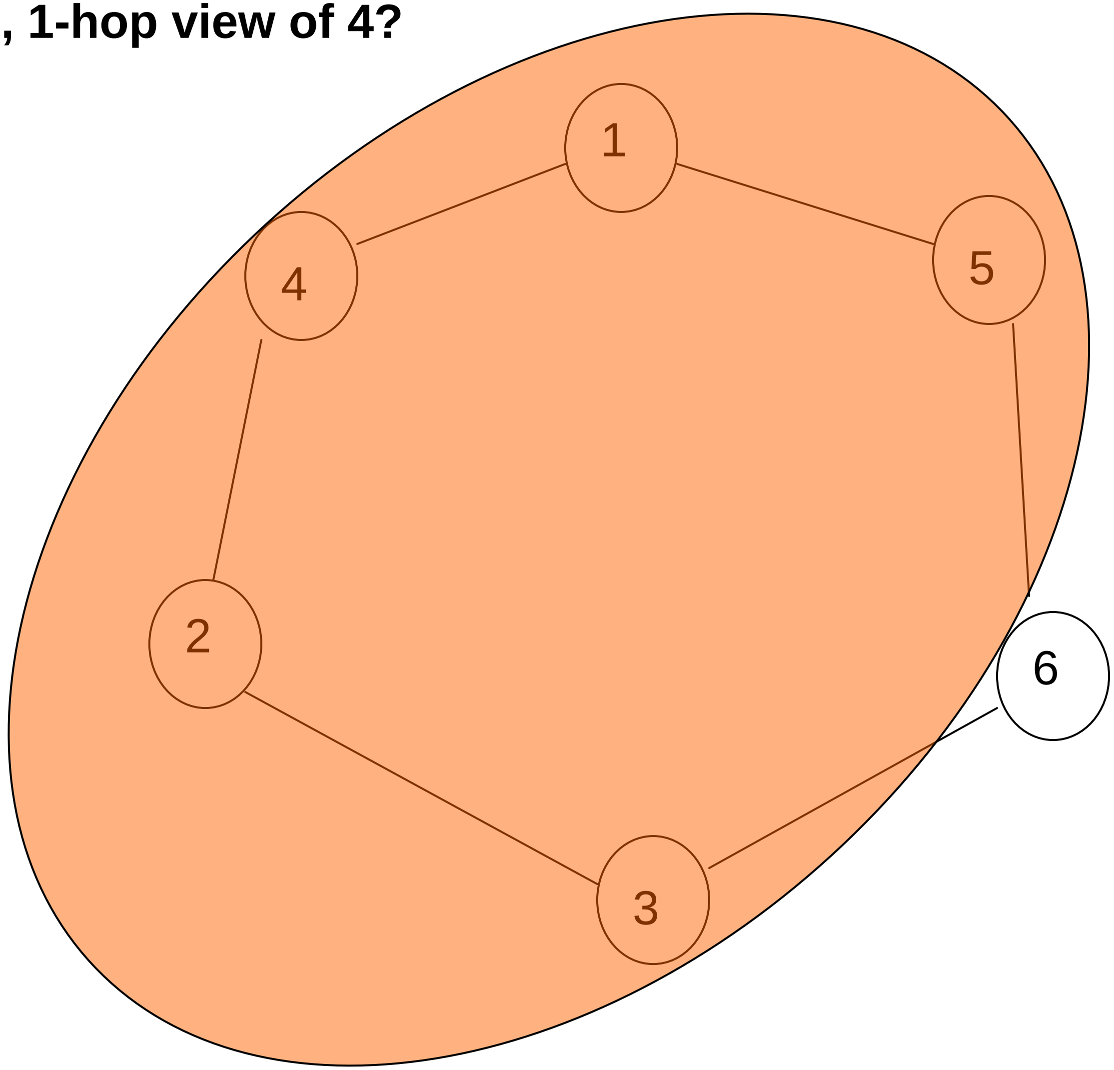


Ring Colorings

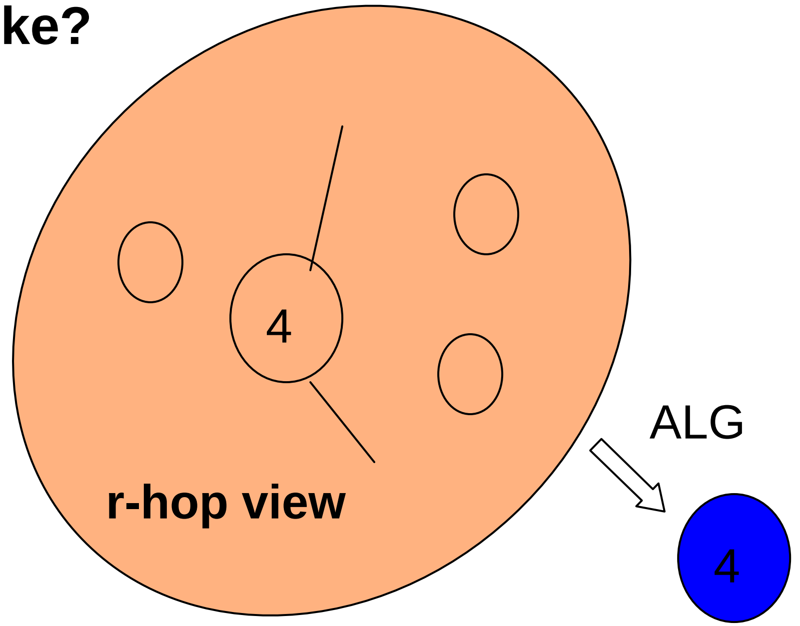
How do r-hop views of our rings look like?

Generally:

The r-hop view of a ring is a (2r+1) tuple:

$$(\mathbf{l}_{-r}, \mathbf{l}_{-r+1}, \ldots, \mathbf{l}_0, \ldots, \mathbf{l}_r)$$

where l_0 is label of considered node v.



A deterministic coloring algorithm maps these tuples to colors!

Question: why tuple and not set? Sense of orientation! ☺

Ring Colorings

When is a coloring valid?

Consider two r-hop views:

$$(l_{-r}, l_{-r+1}, \ldots, l_0, \ldots, l_r)$$

and

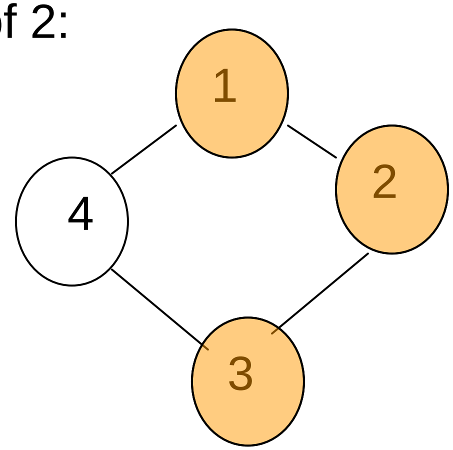
$$(l'_{r}, l'_{r+1}, \ldots, l'_0, \ldots, l'_r)$$

where $l'_i = l_{i+1}$ for $-r \le i \le r-1$ and $l'_r \ne l_i$ for $-r \le i \le r$, so what?

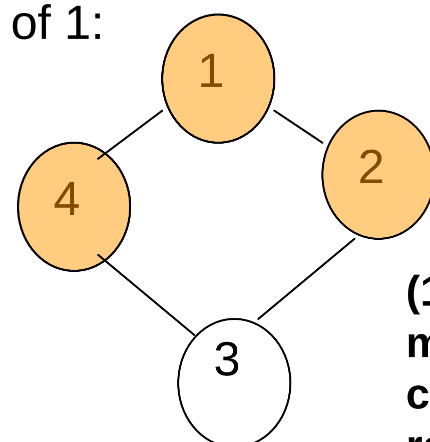
Then the two views can originate from adjacent nodes in the ring! So?

So every algorithm needs to assign different colors to the two views!

1-hop view of 2:



1-hop view of 1:



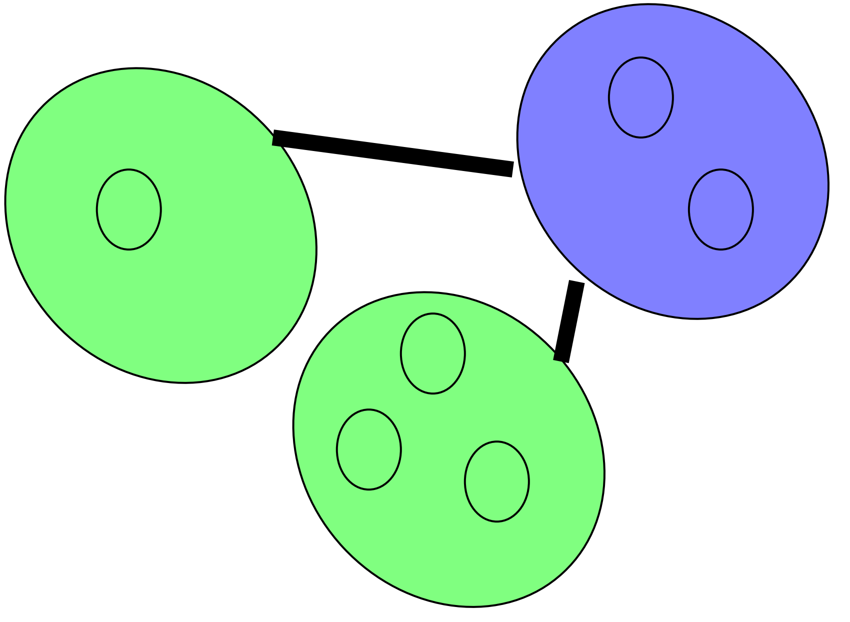
(1,2,3) and (4,1,2) must give different colors (for 1 and 2, respectively!)

Neighborhood Graphs?

What if we define a neighborhood graph: neighborhoods are nodes, and connected if they are conflicting (i.e., views may originate from two adjacent nodes)?

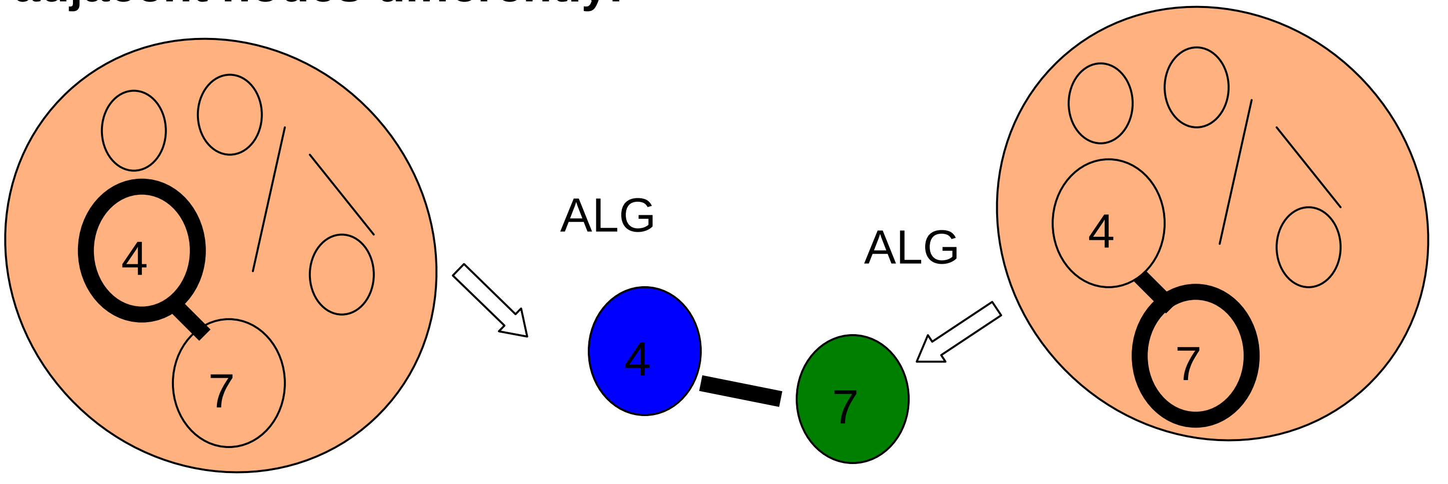
Assume we color the neighborhood graph as follows: „view node" has color of the node the neighborhood is computed from.

How does the coloring of the neighborhood graph look like then?

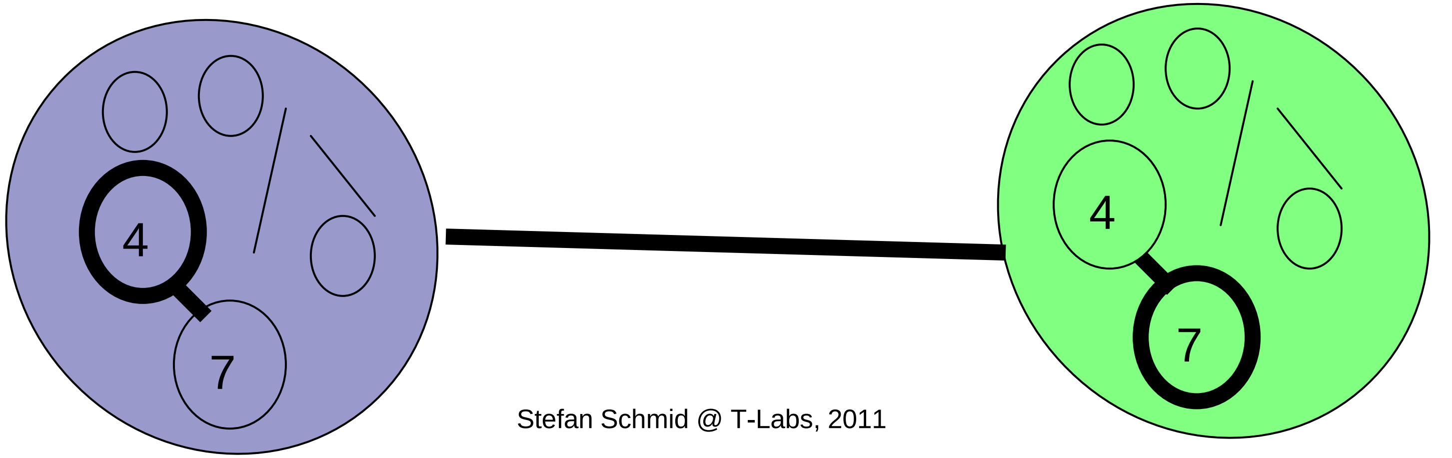


Neighborhood Graphs?

Given collected neighborhoods, canonic coloring ALG colors adjacent nodes differently:



So corresponding views/nodes in neighorhood graph must have different colors too, so valid coloring for neighborhood graph:



Neighborhood Graph

„Formal“ definition:

Neighborhood Graph

The **r-neighborhood graph $N_r(G)$** consists of all r-hop views of G (for all nodes) which are connected iff they could originate from two adjacent nodes.

This lemma motivates the concept:

Lemma

There is an **r-round algorithm** that colors graphs G with c colors iff the chromatic number of the neighborhood graph is $\boldsymbol{\chi(N_r(G)) \leq c}$.

Proof?

Neighborhood Graph

Lemma

There is an **r-round algorithm** that colors graphs G with c colors iff the chromatic number of the neighborhood graph is $\boldsymbol{\chi(N_r(G)) \leq c}$.

Proof:

„=>“: if the chromatic number is larger, there cannot exist a local algorithm with c colors for equivalent real network: all conflicts also in neighborhood graph!

„<=“: We know: local coloring algo is a function that maps r-hop view to color, so to every node of $N_r(G)$...

This coloring is legal: by the definition of r-hop neighborhood graphs, adjacent nodes of $N_r(G)$ must have different colors, since the corresponding nodes in the underlying graph are also adjacent.

(But maybe slightly more than c colors are needed, so“ $\leq$“...

QED

So how do neighborhood graphs of rings look like? How to color them? And how to exploit the lemma to get a lower bound?

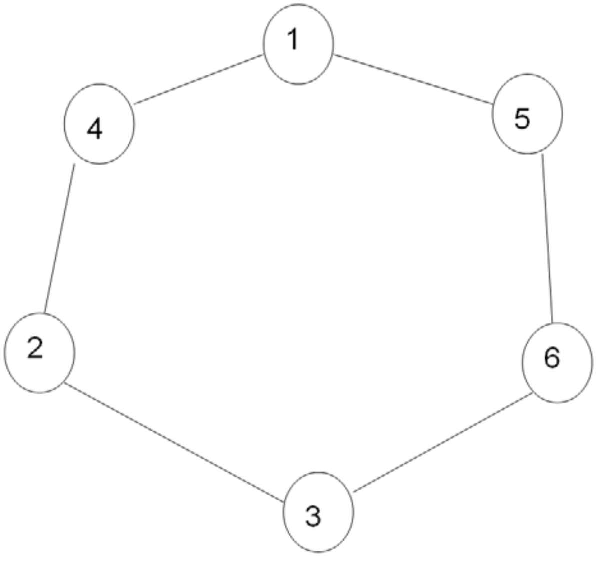
Roadmap

How to find a good lower bound with this lemma?

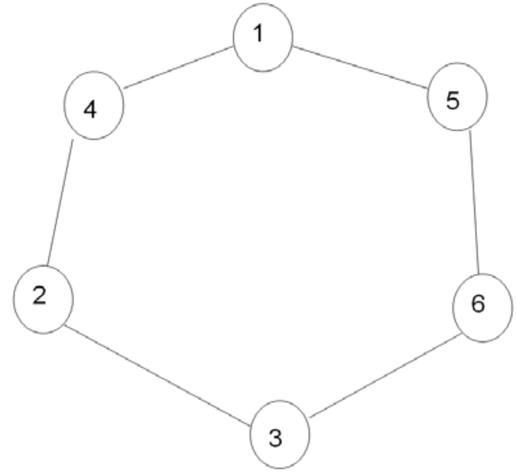
We have to show that $\chi(N_r(G))$ is small only for a large r...

So how does $N_r(G)$ of a ring look like?

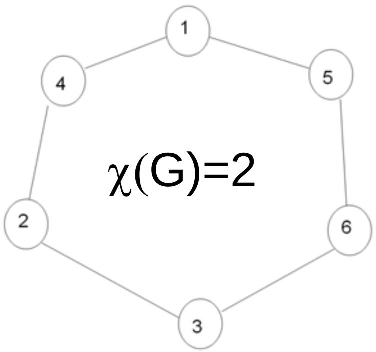
For example of our initial ring graph?



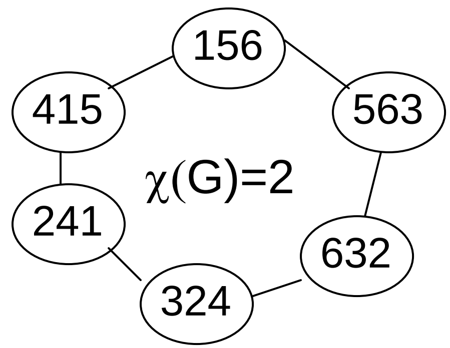
N_r(Given Ring)?



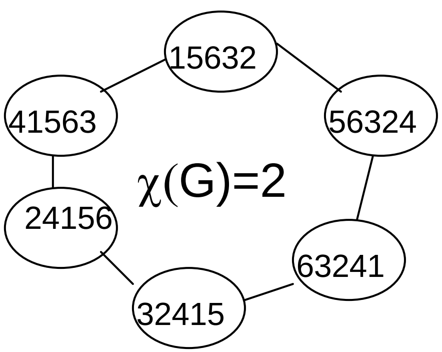
0-hop neighborhood graph?



1-hop neighborhood graph?



2-hop neighborhood graph?



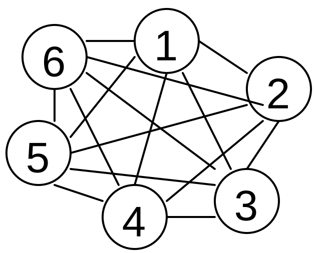
So 0 or 1 round to 2-color?!?

Attention: We are interested in neighborhood graphs of families of graphs / rings! A given graph is easy! ☺

N_r(Ring)?

r-hop neighborhood graph for ring family (n=6 known)?

$$\mathbf{N_0 = ?}$$

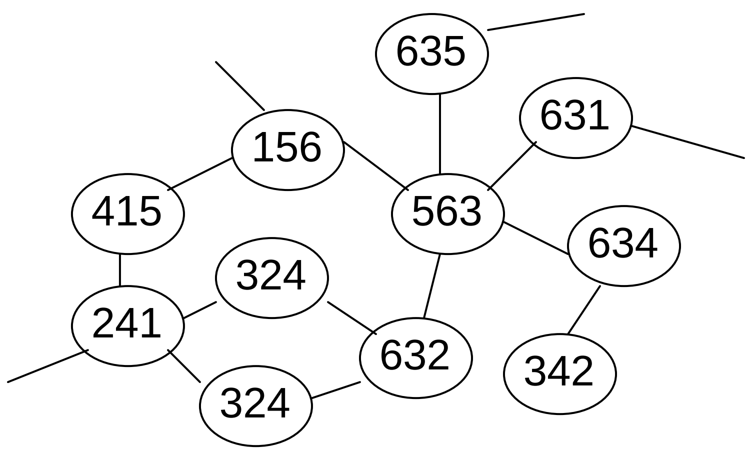


Complete graph: every node could be neighbor of every other node

$$\chi(\mathbf{N_0}) = ?$$
$$= n$$

Any 0-local algorithm can only choose its ID as a color...: n colors

$$\mathbf{N_1 = ?}$$

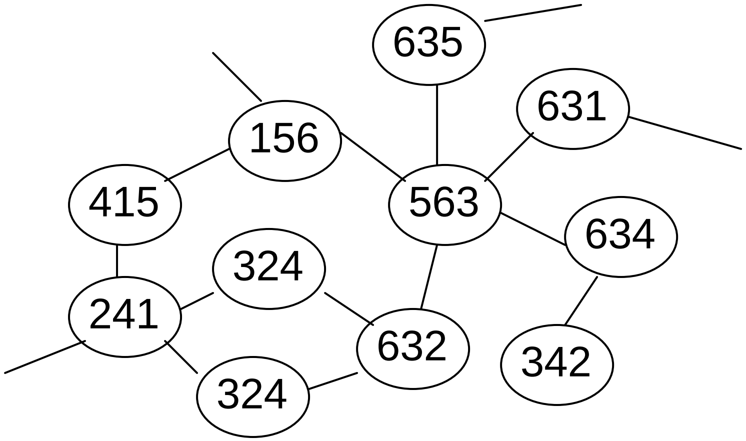


$$\chi(\mathbf{N_1}) = ???$$

N_r(Ring)?

What happens for larger neighborhoods?

Intuitively, the larger the considered neighborhood, the less conflicts are possible! Chromoatic number declines for larger r... (We will see: in logarithmic factors „per hop“!)
At some point, the graph family member is clear!



What if ring size is unknown?

We can see it as disconnected or different neighborhood graphs for different n. For ring not much of a difference because we are interested in neighborhoods much smaller than n anyway...

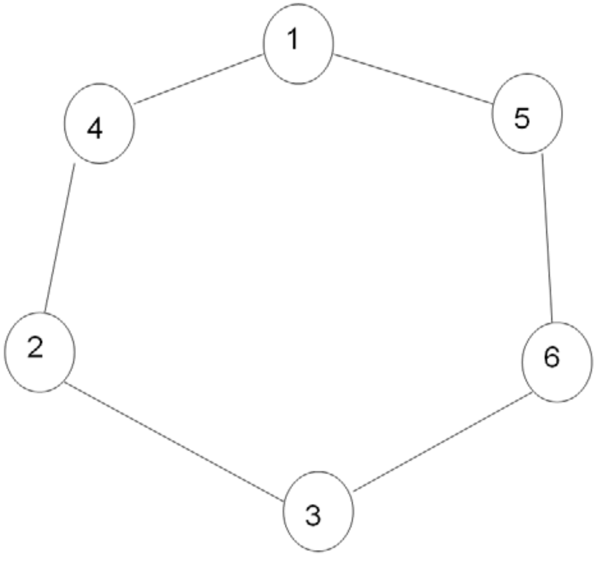
Roadmap

Main question now: What is $\chi(N_r(\text{Ring}))$??

Difficult... So let‘s focus on a graph which is similar, but has less conflicts and hence its chromatic number can be used instead for the lower bound!

What graphs are good then?

E.g., subgraphs...: less conflicts, so weaker lower bound when applying our lemma!



Neighborhood Graph of Ring

Instead of defining neighborhood graphs for rings:

$B_{k,n}$ Graph

Assume two integers k,n where $n \geq k$. The $\mathbf{B}_{k,n}$ graph consists of the nodes of k-tuples of increasing node labels (from {1,...,n}). There is a directed edge from node α to node β iff $\forall\, i \in \{1,...,k\text{-}1\}$: $\boldsymbol{\beta_i = \alpha_{i+1}}$.

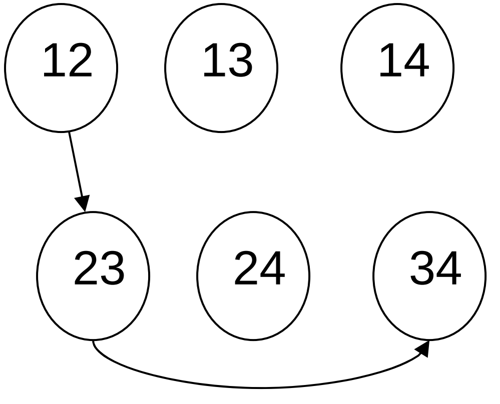
Example: k=2, n=4

$V(B_{k,n})$ = ?

= {(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)}

$E(B_{k,n})$ = ?

= {((1,**2**),(**2**,3)), ((2,**3**),(**3**,4))}



Neighborhood Graph of Ring

What does this have to do with rings?!

Lemma

Viewed as an undirected graph, $\mathbf{B_{2r+1,n}}$ is a subgraph of the r-neighborhood graph of n-node rings with node labels from {1,...,n}.

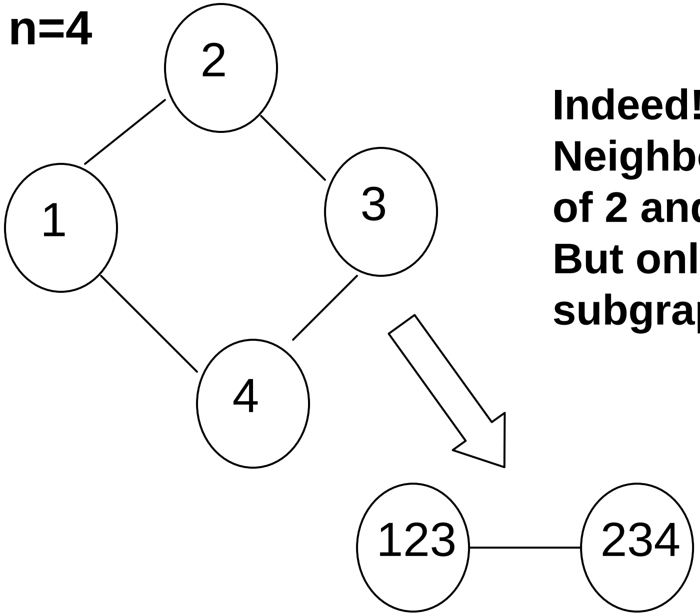
Example: Neighborhood r=1 (so k=3), n=4

$V(B_{k,n}) = ?$

$= \{(1,2,3),(2,3,4)\}$

$E(B_{k,n}) = ?$

$= \{\{(1,\mathbf{2},3),(\mathbf{2},3,4)\}\}$



Indeed! Neighborhood of 2 and 3! But only a subgraph! (Why?)

Neighborhood Graph of Ring

Lemma

Viewed as an undirected graph, $\mathbf{B_{2r+1,n}}$ is a subgraph of the r-neighborhood graph of n-node rings with node labels from {1,...,n}.

Proof?

The set of k-tuples of increasing labels is a subset of all the k-tuples / nodes (in our example, views of node 1 and 4 are missing)**.**

Two nodes are only connected in $B_{2r+1,n}$ if there is also an edge in the neighborhood graph (because labels are ordered, the views must come from adjacent nodes): not more edges/conflicts.

QED

What does it mean?!

Chromatic number of $B_{2r+1,n}$ good for lower bound of our problem!

- We have to compute lower bound for $\chi(B_{2r+1,n})$!
- How? With another helper graph... ☺

Helper Graph

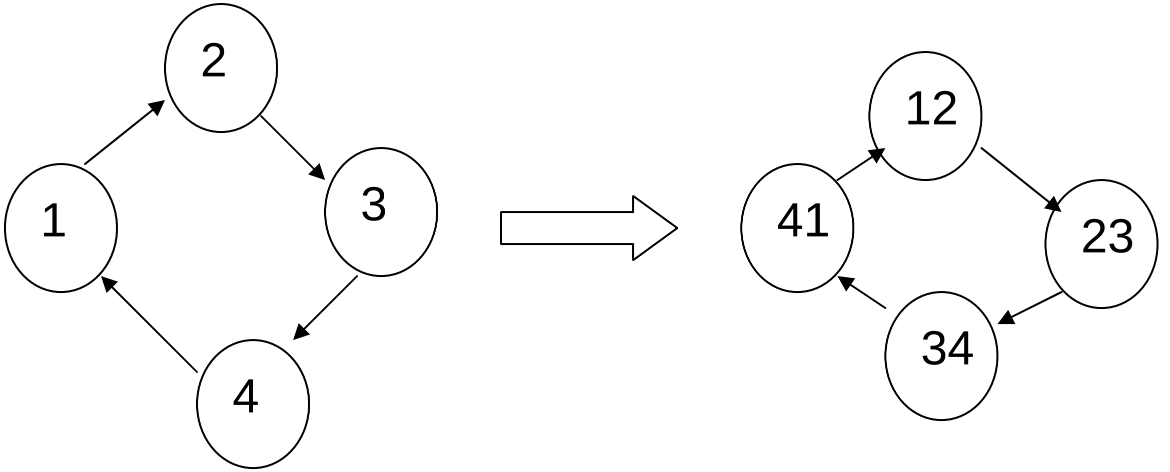
The following graph is helpful to analyze $B_{2r+1,n}$: What does it mean?

Diline Graph

The directed line graph (diline graph) **DL(G)** of a directed graph G=(V,E) is defined as follows: V(DL(G))=E, and there is a directed edge ((w,x),(y,z)) iff x=y.

In other words: DL(G) consists of the node representing the edges of G, and two nodes are connected if the corresponding edges „follow“ after each other.

Example:



What is the relation to $B_{k,n}$?!

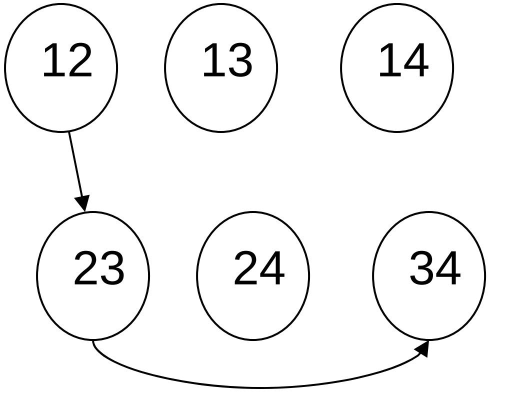
Recursive Construction

$B_{k,n}$ can be recursively defined by directed line graphs!

Lemma

$B_{k+1,n} = DL(B_{k,n})$

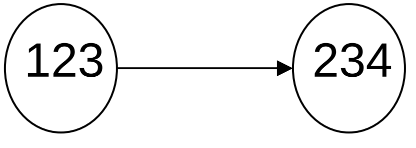
Really? Example: k=2, n=4?



$V(B_{k,n}) = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

$E(B_{k,n}) = \{((1,\mathbf{2}),(\mathbf{2},3)), ((2,\mathbf{3}),(\mathbf{3},4))\}$

Example: k=3, n=4?



$V(B_{k,n}) = \{(1,2,3),(2,3,4)\}$

$E(B_{k,n}) = \{((1,\mathbf{2},\mathbf{3}),(\mathbf{2},\mathbf{3},4))\}$

Recursive Construction

$B_{k,n}$ can be recursively defined by directed line graphs!

Lemma

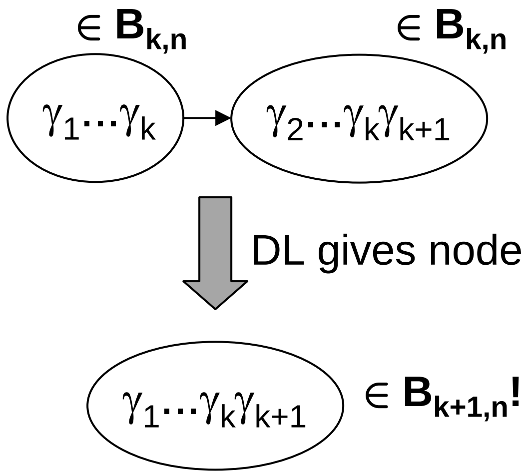
$B_{k+1,n} = DL(B_{k,n})$

Proof?

By the definition of $B_{k,n}$, two nodes α, β are connected if the first k-1 labels in β are the same as the last k-1 labels of α.

Therefore, the pair (α,β) can be represented by a (k+1) tuple $\gamma=(\gamma_1,\ldots,\gamma_{k+1})$ with $\gamma_1=\alpha_1$, $\gamma_i=\beta_{i-1}=\alpha_i$ for $2 \leq i \leq k$, and $\gamma_{k+1}=\beta_k$. The labels of γ are increasing too! So $B_{k+1,n}$ has the same node set as $DL(B_{k,n})$.

What about the edges?



Recursive Construction

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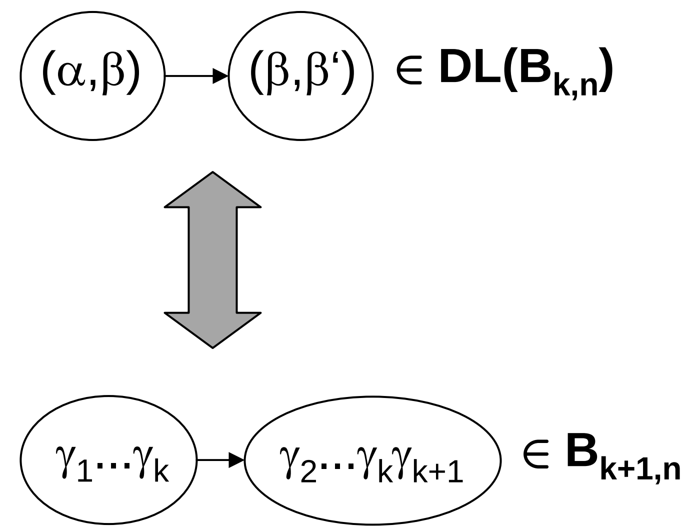


Proof (continued for edges...)

There is an edge between two nodes (α,β) and (α',β') of $DL(B_{k,n})$ if $\beta=\alpha'$.
This is equivalent to that the two corresponding $(k+1)$-tuples γ and γ' are neighbors in $B_{k+1,n}$: the last k labels of γ are equivalent to the first k labels of γ'.

QED

So, $B_{k,n}$ graphs are simply „iterated line graphs“!

Chromatic Numbers

Implication for colorings, coloring G vs DL(G)?

Proof idea?

Lemma

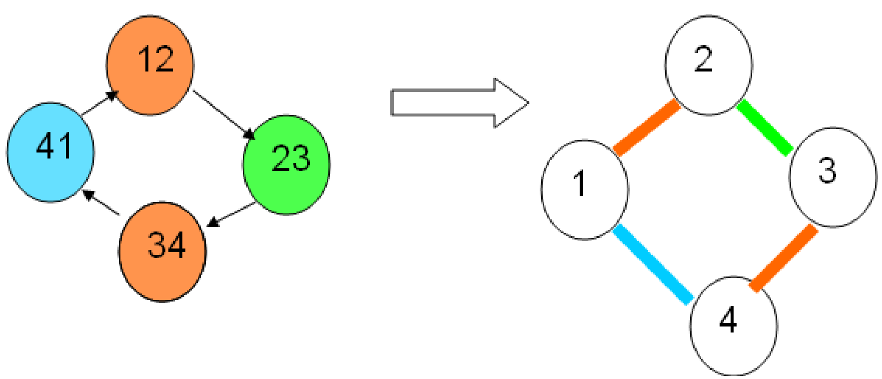
$\chi(DL(G)) \geq \log_2(\chi(G))$

Given a c-coloring of DL(G) we construct a **2^c coloring of G** (so minimal coloring of G can only be smaller).

How does coloring of G and DL(G) relate?

Note: A c-coloring of the diline graph DL(G) can be seen as a coloring of the edges of G such that no two adjacent edges have the same color (definition of DL(G)).

DL(G):



: G

Attention: these graphs are generally not rings of course! ☺

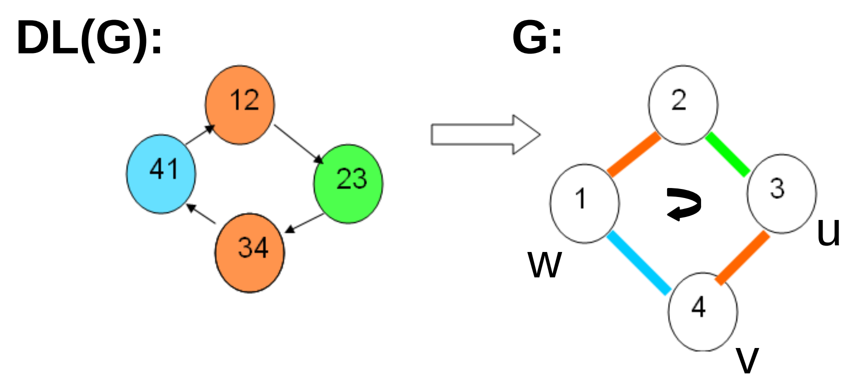
Chromatic Numbers

Implication for colorings, coloring G vs DL(G)?

Lemma

$\chi(DL(G)) \geq \log_2(\chi(G))$

Proof idea (continued...)



For a node $v \in G$, let $\mathbf{S_v}$ denote the set of colors of its outgoing edges in the ring. Let (u,v) be a directed edge in G and let x be the color of (u,v). Thus: $x \in S_u$.

No edge (v,w) can have color x, so $x \notin S_v$, so $\mathbf{S_u \neq S_v}$: neighboring nodes in G must have different „out-edge-color-sets“!
We can use these color sets S to obtain a vertex coloring of G: the color of a node u is S_u. This coloring must be legal!
As we can have at most 2^c subsets (of c vertex colors of DL(G) and hence edge colors of G), the coloring has at most 2^c colors.

QED

Chromatic Numbers

Chromatic number of $B_{k,n}$?

Recall: Gives lower bound for r-hop coloring algo!
Intuitively: Each time the local view is increased, the chromatic number goes down at most by a log factor!

Lemma

$$\chi(B_{1,n}) = n \text{ and } \chi(B_{k,n}) \geq \log^{(k-1)} n$$

Proof idea?

$B_{1,n}$ is the complete graph.
For larger k, it holds by induction due to our lemmas!

QED

Finally: Lower Bound

Combining everything gives our lower bound! ☺

Lower Bound

Any deterministic distributed algorithm to color a ring with 3 or less colors needs at least $(\log^* n)/2-1$ rounds.

Proof idea?

We need to show that $\chi(B_{2r+1,n})>3$ for all $r<(\log^* n)/2-1$.
We know that $\chi(B_{2r+1,n}) \geq \log^{(2r)} n$.
And $B_{2r+1,n}$ is subgraph of neighborhood graph we actually want!
The rest is simple maths...

QED

Literature for further reading:

- Peleg‘s book (as always ☺)

End of lecture