1 Pumping Lemma [Exam]

The Pumping Lemma in a Nutshell
Given a language $L$, assume for contradiction that $L$ is regular and has the pumping length $p$. Construct a suitable word $w \in L$ with $|w| \geq p$ ("there exists $w \in L$") and show that for all divisions of $w$ into three parts, $w = xyz$, with $|x| \geq 0$, $|y| \geq 1$, and $|xy| \leq p$, there exists a pumping exponent $i \geq 0$ such that $w' = xy^iz \notin L$. If this is the case, $L$ is not regular.

Language $L_1$ can be shown to be non-regular using the pumping lemma. Assume for contradiction that $L_1$ is regular and let $p$ be the corresponding pumping length. Choose $w$ to be the word $0110^p1$. Because $w$ is an element of $L_1$ and has length more than $p$, the pumping lemma guarantees that $w$ can be split into three parts, $w = xyz$, where $|xy| \leq p$ and for any $i \geq 0$, we have $xy^iz \in L_1$. In order to obtain the contradiction, we must prove that for every possible partition into three parts $w = xyz$ where $|xy| \leq p$, the word $w$ cannot be pumped. We therefore consider the various cases.

a) If $y$ starts anywhere within the first three symbols (i.e. $011$) of $w$, deleting $y$ (pumping with $i = 0$) creates a word with an illegal prefix (e.g. $10^p1^p$ for $y = 01$).

b) If $y$ consists of only 0s from the second block, the word $w' = xy^2z$ has more 0s than 1s in the last $|w'| - 3$ symbols and hence $c \neq d$.

Note that $y$ cannot contain 1s from the second block because of the requirement $|xy| \leq p$.

We have shown that for all possible divisions of $w$ into three parts, the pumped word is not in $L_1$. Therefore, $L_1$ cannot be regular and we have a contradiction.

Be Careful!
The argumentation above is based on the closure properties of regular languages and only works in the direction presented. That is, for an operator $\diamond \in \{\cup, \cap, \cdot\}$, we have:

If $L_1$ and $L_2$ are regular, then $L = L_1 \diamond L_2$ is also regular.

If either $L_1$ or $L_2$ or both are non-regular, we cannot deduce the non-regularity of $L$ or vice-versa. Moreover, $L$ being regular does not imply that $L_1$ and $L_2$ are regular as well. This may sound counter-intuitive which is why we give examples for the three operators.

• $L = L_1 \cup L_2$: Let $L_1$ be any non-regular language and $L_2$ its complement. Then $L = \Sigma^*$ is regular.
• \( L = L_1 \cap L_2 \): Let \( L_1 \) be any non-regular language and \( L_2 \) its complement. Then \( L = \emptyset \) is regular.

• \( L = L_1 \cup L_2 \): Let \( L_1 = \{ a^p \} \) (a regular language) and \( L_2 = \{ a^{p \ast} \mid p \text{ is prime} \} \) (a non-regular language) then \( L = \{ aaa^* \} \) is regular.

Hence, to prove that a language \( L_x \) is non-regular, you assume it to be regular for contradiction. Then you combine it with a regular language \( L_y \) to obtain a language \( L = L_x \cup L_y \). If \( L \) is non-regular, \( L_x \) could not have been regular either.

### 2 Deterministic Finite Automata [Exam]

We could use the systematic transformation scheme presented in the lecture (slide 1/75). Considering the large number of states, however, this will easily lead to an explosion of states in the derandomized automaton. Hence, we build the deterministic finite automaton in a step-wise manner, only creating those states that are actually required: Initially, the automaton requires a 0. Subsequently, only a 1 is accepted. Including the various transitions, this 1 can lead to three different states, namely states 2, 3, and 4.

In any of the states 2, 3, and 4, only a 1 is accepted. Assume that the automaton is currently in state 2, this 1 can lead to states \( \{ 2, 3, 4 \} \) when including all \( \varepsilon \)-transitions. When in state 3, the 1 leads to states \( \{ 2, 3, 4, 5 \} \) and finally, when being in state 4, the reachable states given a 1 are \( \{ 2, 3, 4 \} \). Hence, a 1 leads from state \( \{ 2, 3, 4 \} \) to state \( \{ 2, 3, 4, 5 \} \). Repeating the same process for state \( \{ 2, 3, 4, 5 \} \), we can see that, again, only a 1 is accepted, which leads to state \( \{ 2, 3, 4, 5, 6 \} \). Because the state 6 in the original NFA was an accepting state, \( \{ 2, 3, 4, 5, 6 \} \) is also accepting in the DFA. From state \( \{ 2, 3, 4, 5, 6 \} \), an additional 1 will lead to another accepting state \( \{ 1, 2, 3, 4, 5, 6 \} \). And from this state, any subsequent 1 returns to state \( \{ 1, 2, 3, 4, 5, 6 \} \) as well.

What happens if a 0 occurs in the input? This is feasible only when the deterministic state includes either state 1 or state 6. In state \( \{ 2, 3, 4, 5, 6 \} \), a 0 necessarily leads to state \( \{ 4 \} \), whereas in state \( \{ 1, 2, 3, 4, 5, 6 \} \) a 0 leads to state \( \{ 2, 4 \} \). In both of these states, the only acceptable input symbol is a 1 and leads to the state \( \{ 2, 3, 4 \} \). Hence, the deterministic finite automaton looks like this:
It can easily be seen, that first the states \{4\}, \{2, 4\} and then the states \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\} can be merged and hence, the automaton can be reduced to the one shown in the next figure.

This is not a DFA yet, because the crash state is still missing. The final deterministic automaton looks like this:

\[
\begin{array}{cccccc}
  & 0 & 1 & 0 & 1 & 1 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} & \\
\end{array}
\]

\[
\begin{array}{cccccc}
  & 0,1 & 0 & 1 & 0 & 1 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} & \\
\end{array}
\]

\[
\begin{array}{cccc}
  & 0 & 1 & 0 & 1 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} \\
\end{array}
\]

\[
\begin{array}{cccc}
  & 0 & 1 & 0 & 1 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} \\
\end{array}
\]

3 Transforming Automata [Exam]

The regular expression can be obtained from the finite automaton using the transformation presented in the script on slide 1/85. After ripping out state \(q_2\), the corresponding GNFA looks like this:

After also removing state \(q_1\), the GNFA looks as follows.

\[
\begin{array}{cccc}
  & 0,1^*0 & 0 & 0 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} \\
\end{array}
\]

\[
\begin{array}{cccc}
  & 0 & 0,1^*0 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} \\
\end{array}
\]

\[
\begin{array}{cccc}
  & 0 & 0,1^*0 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} \\
\end{array}
\]

\[
\begin{array}{cccc}
  & 0 & 0,1^*0 \\
\{1\} & \{2, 4\} & \{2, 3, 4\} & \{2, 3, 4, 5\} & \{1, 2, 3, 4, 5, 6\} \\
\end{array}
\]

Eliminating the last state \(q_3\) yields the final solution, which is \((01^*0)^*1(0 \cup 11^*0(01^*0)^*1)^*\).

Note: Ripping out the interior states in a different order yields a distinct yet equivalent regular expression. The order \(q_3, q_2, q_1\), for example, results in \(((0 \cup 10^*1)1^*0)^*10^*\).

4 Regular and Context-Free Languages

a) Sometimes, even simple grammars can produce tricky languages. We can interpret the 1s and 2s of the second production rule as opening and closing brackets. Hence, \(L(G)\) consists of all correct bracket terms where at least one 0 must be in each bracket.
Choose \( w = 1^p02^p \in L(G) \). Let \( w = xyz \) with \( |xy| \leq p \) and \( |y| \geq 1 \) (pumping lemma). Because of \( |xy| \leq p \), \( xy \) can only consist of 1s. According to the pumping lemma, we should have \( xy^iz \in L \) for all \( i \geq 0 \). However, by choosing \( i = 0 \) we delete at least one 1 and get a word \( w' = 1^{p-|y|}02^p \) with \( |y| \geq 1 \). \( w' \) is not in \( L \) since it has fewer 1s than 2s. This means that \( w \) is not pumpable and hence, \( L(G) \) is not regular.

b) Since every regular language is also context-free, we can choose an arbitrary regular language. For example, we can choose the language \( L = \{0^n1, n \geq 1 \} \) which is clearly regular. A context-free grammar for this language uses only the production \( S \rightarrow 0S | 1 \).