An application of the Cole–Vishkin algorithm: approximating vertex covers in anonymous networks

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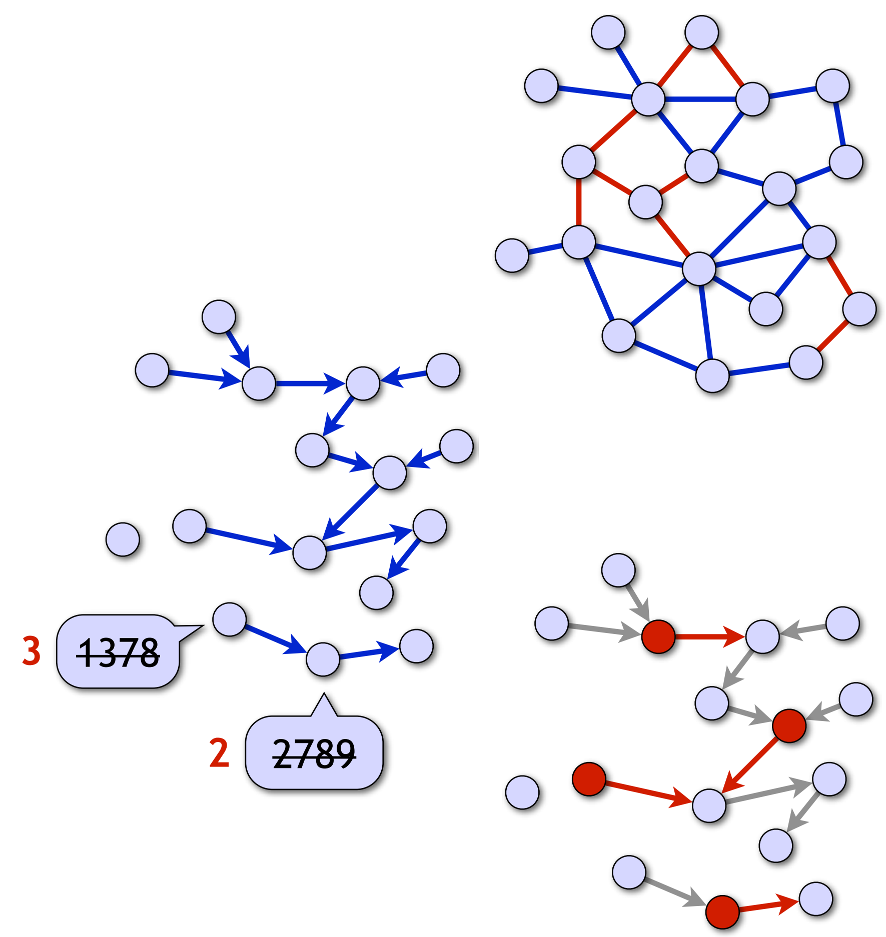
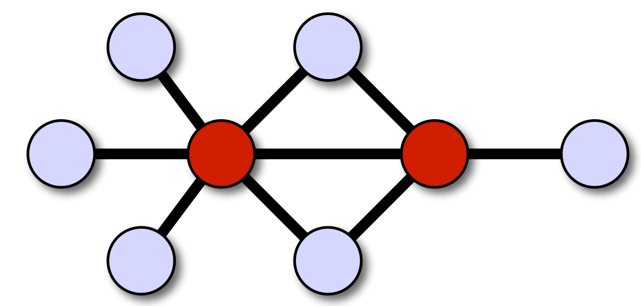
Principles of Distributed Computing

10 March 2010



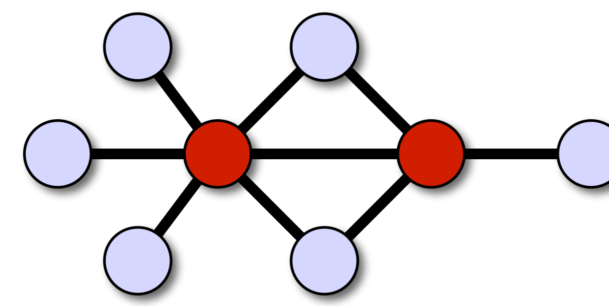
Vertex cover problem



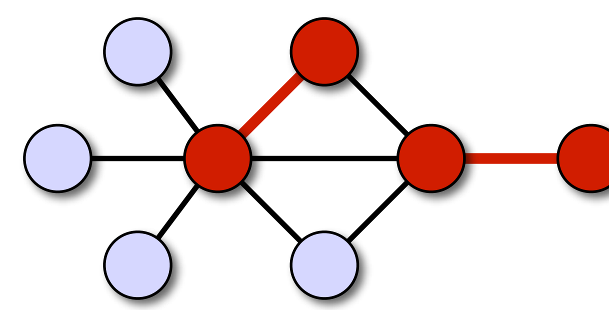
- **Vertex cover** for a graph *G*:
  - Subset *C* of nodes that "covers" all edges: each edge incident to at least one node in *C*
- Minimum vertex cover:
  - Vertex cover with the smallest number of nodes
- Minimum-weight vertex cover:
  - Vertex cover with the smallest total weight

Vertex cover problem

- Classical NP-hard optimisation problem: given a graph *G*, find a minimum vertex cover



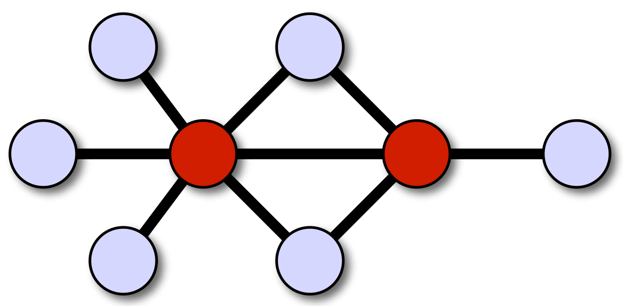
- Simple 2-approximation algorithm:
  - Find a **maximal matching**, output all endpoints
  - At most 2 times as large as minimum VC



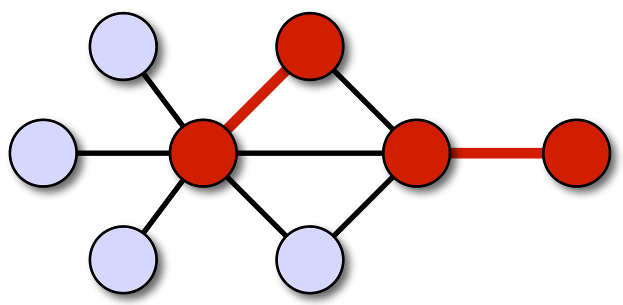
- No polynomial-time algorithm with approximation factor 1.9999 known

Research question

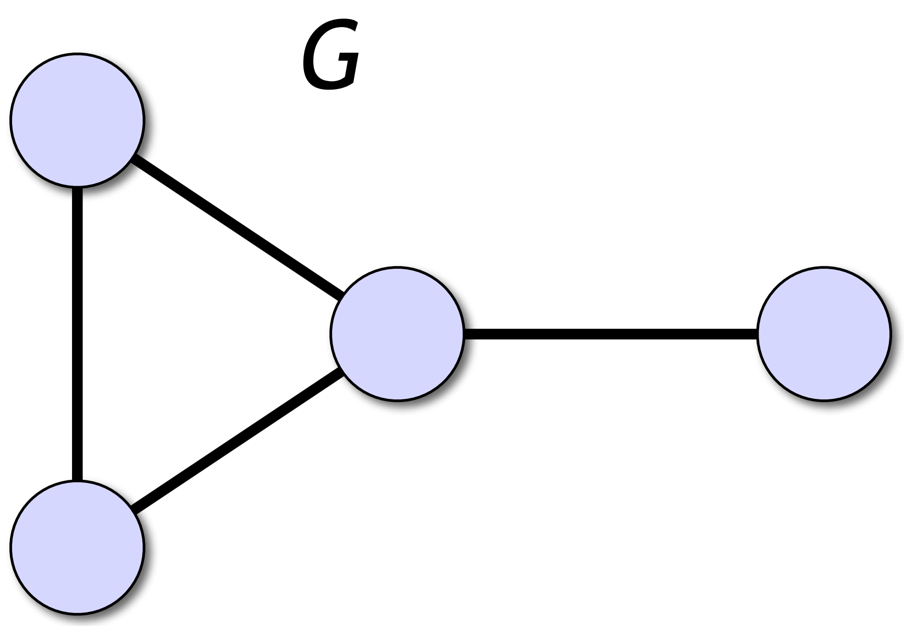
- Can we find a 2-approximation of a minimum vertex cover in a **distributed setting**?



- Focus:
    - Fast, synchronous, **deterministic** distributed algorithms
    - **Port-numbering model**

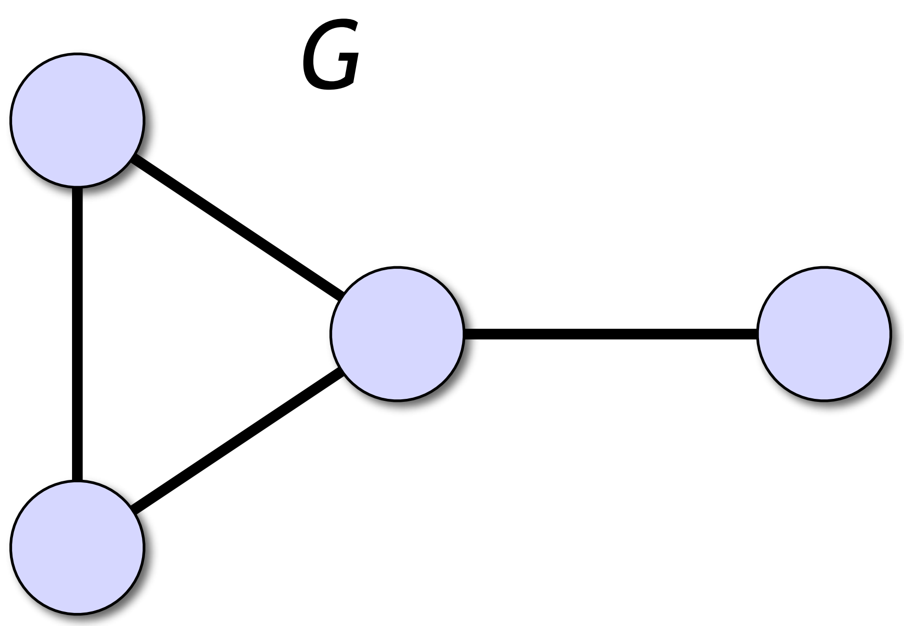


Distributed algorithms



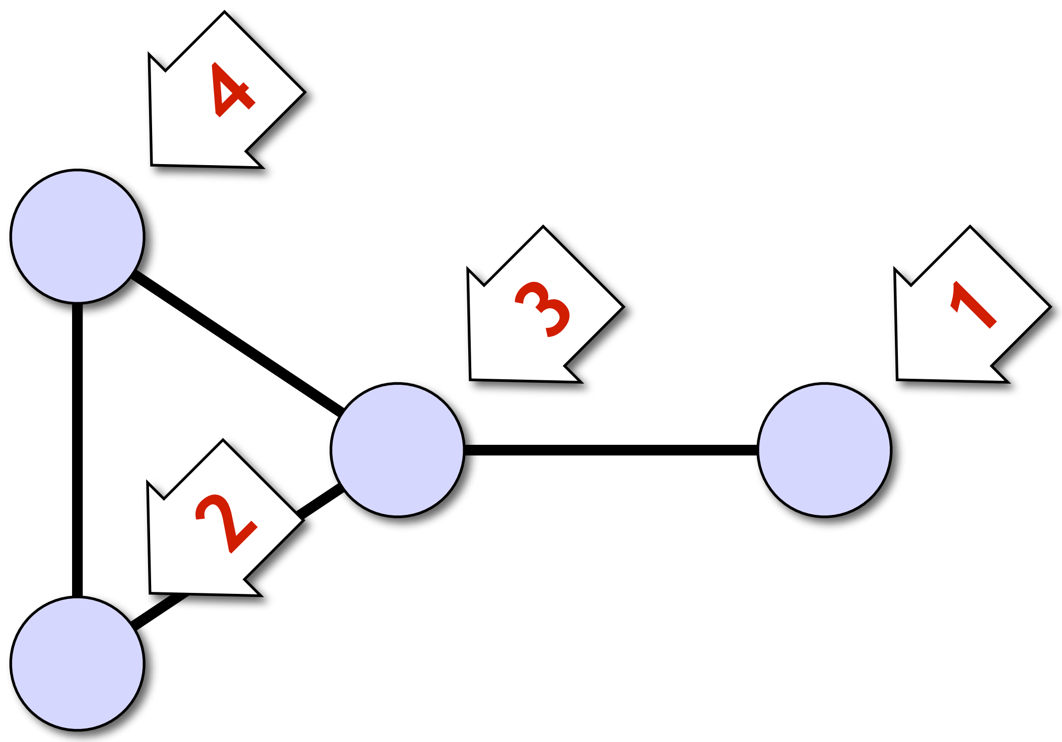
- Communication graph *G*
- Node = computer
- Edge = communication link

Distributed algorithms



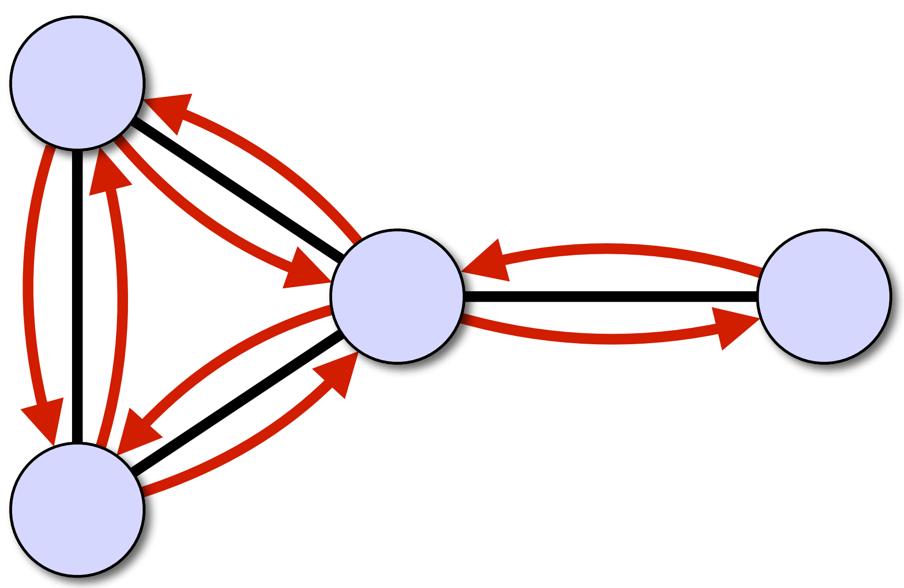
- All nodes are identical, run the same algorithm
- ***We*** can choose the algorithm
- An ***adversary*** chooses the structure of *G*
- Our algorithm must produce a valid vertex cover in any graph *G*

Synchronous distributed algorithms



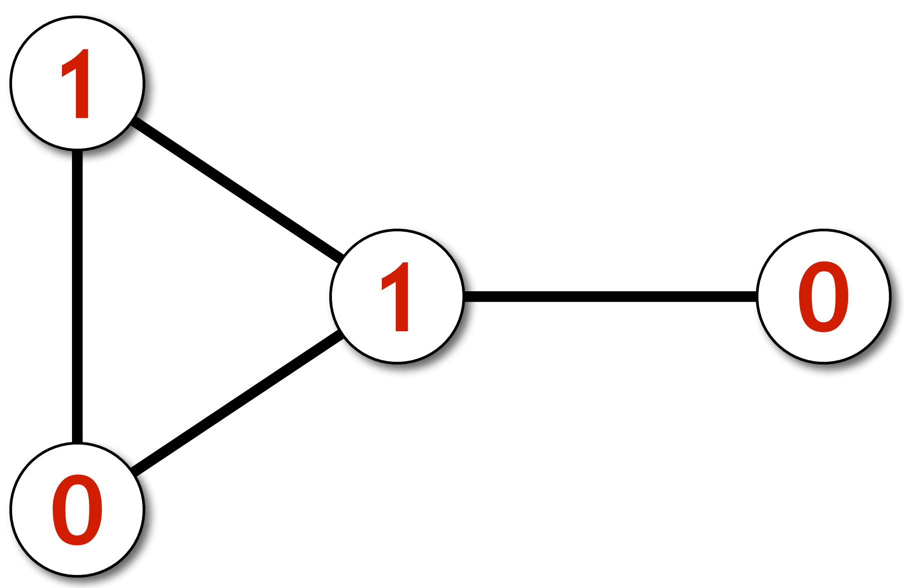
1. Each node reads its own **local input**:
    - node identifier
        - if we assume unique node IDs
    - node weight
        - if we study weighted graphs

Synchronous distributed algorithms



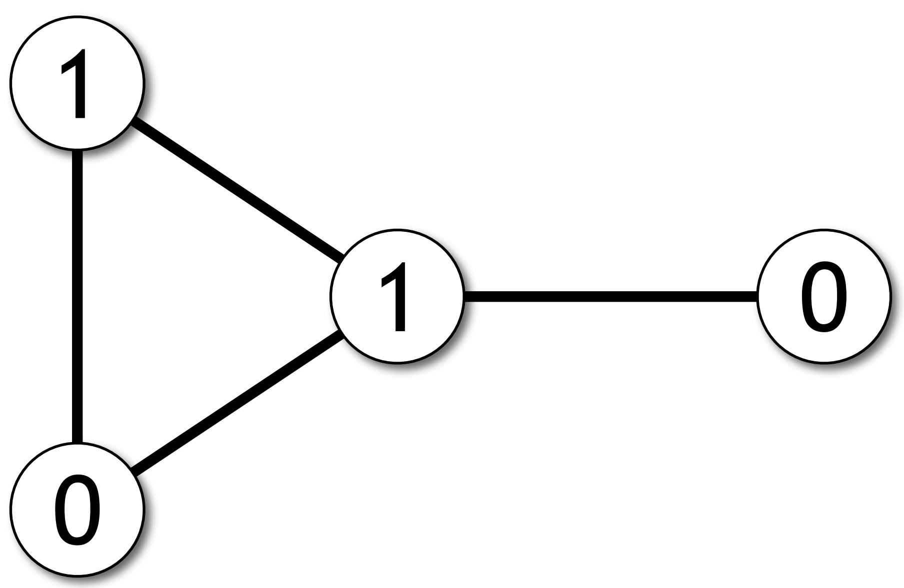
1. Each node reads its own **local input**
2. Repeat synchronous communication rounds
   …

Synchronous distributed algorithms



1. Each node reads its own **local input**
2. Repeat synchronous **communication rounds** until all nodes have announced their local outputs
   - 1 = in vertex cover

Synchronous distributed algorithms

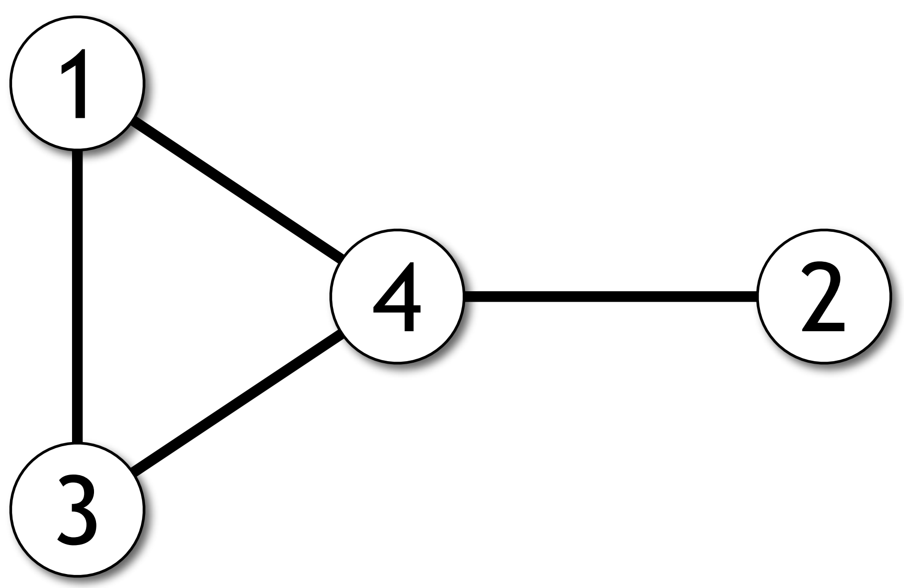


- Running time = **number of rounds**
- Worst-case analysis

Distributed algorithms: two models

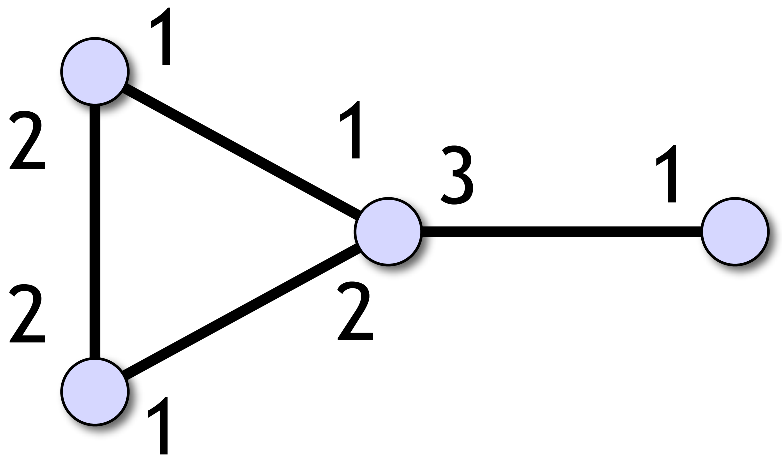
1. Unique identifiers
    - The standard model commonly used in the field
2. Port-numbering model
    - Much weaker model of computation
    - Our focus today

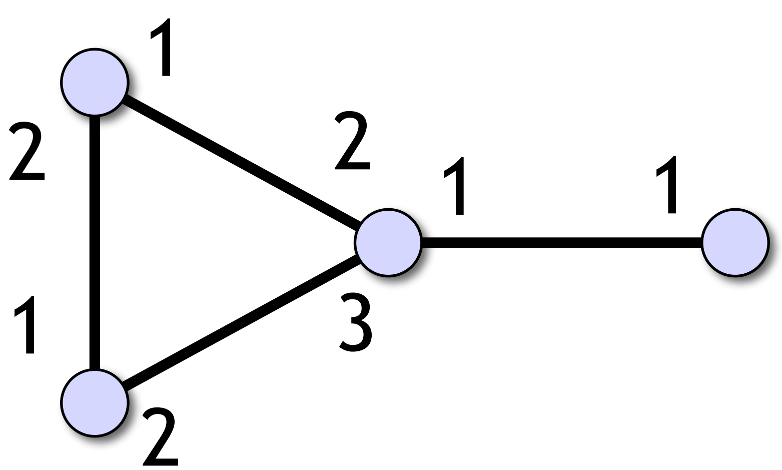
Model 1: Unique identifiers



- Node identifiers are a permutation of 1, 2, …, n
  - Or a subset of 1, 2, …, poly(n)
- Permutation chosen by adversary

Model 2: Port-numbering model





- No unique identifiers
- A node of degree *d* can refer to its neighbours by integers 1, 2, ..., *d*
- Port-numbering chosen by adversary

Cole–Vishkin algorithm

Chapter 1: algorithms ***"6-Color"*** and ***"Six-2-Three"***

- **Colour reduction technique**
  - For cycles and trees — similar ideas can be used in more general graphs as well
- Replaces a k-colouring with an O(log k)-colouring in one round
  - Repeated application: replaces a k-colouring with a 6-colouring in $O(\log^* k)$ rounds
  - Simple additional tricks can be used to find a 3-colouring

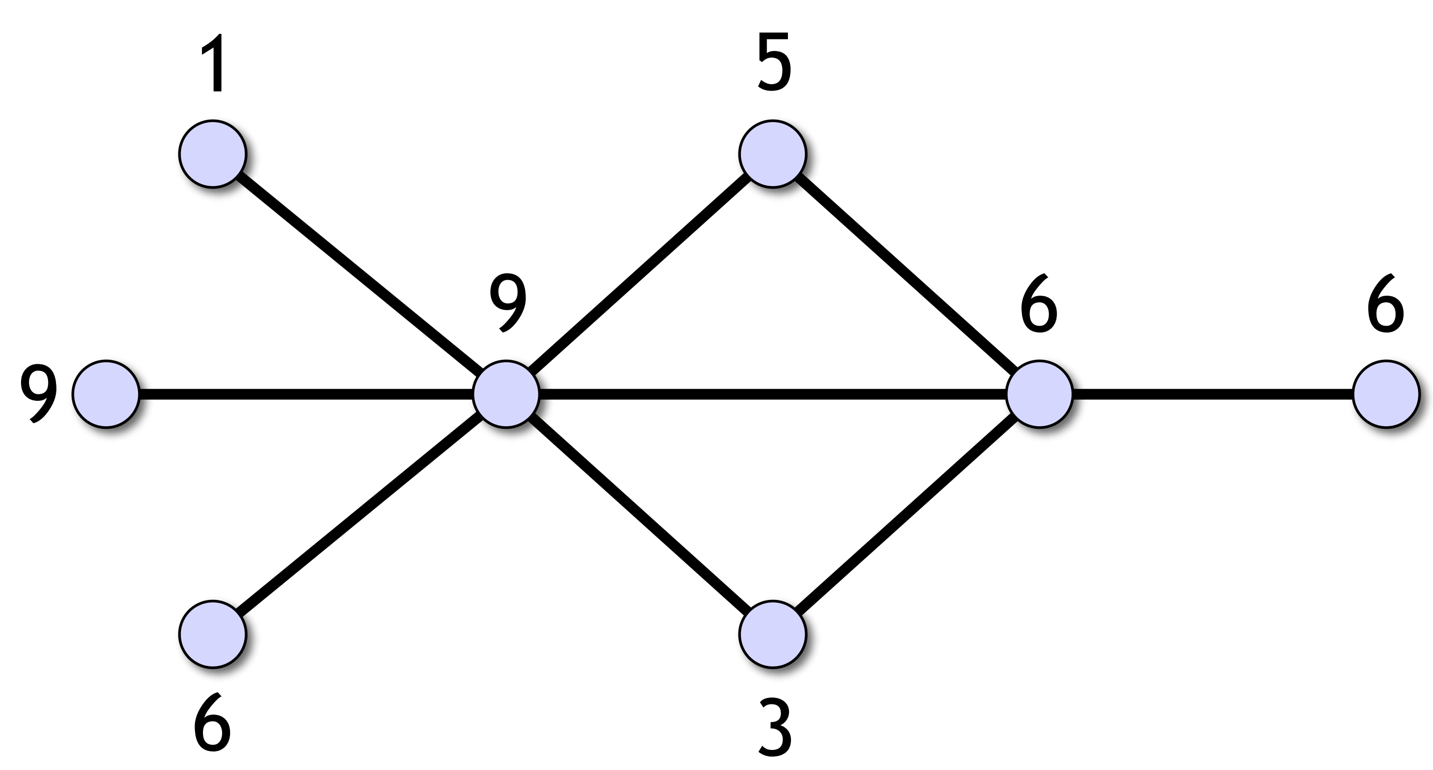
Cole–Vishkin algorithm

Chapter 1: algorithms ***"6-Color"*** and ***"Six-2-Three"***

- Colour reduction technique
- If we have **unique identifiers**:
  - Interpret unique IDs as an n-colouring
  - Cole–Vishkin finds a 3-colouring in $O(\log^* n)$ rounds
- However, we can't use this trick in the **port-numbering model**
  - And we are trying to find a vertex cover, not a colouring!

Vertex cover in the port-numbering model

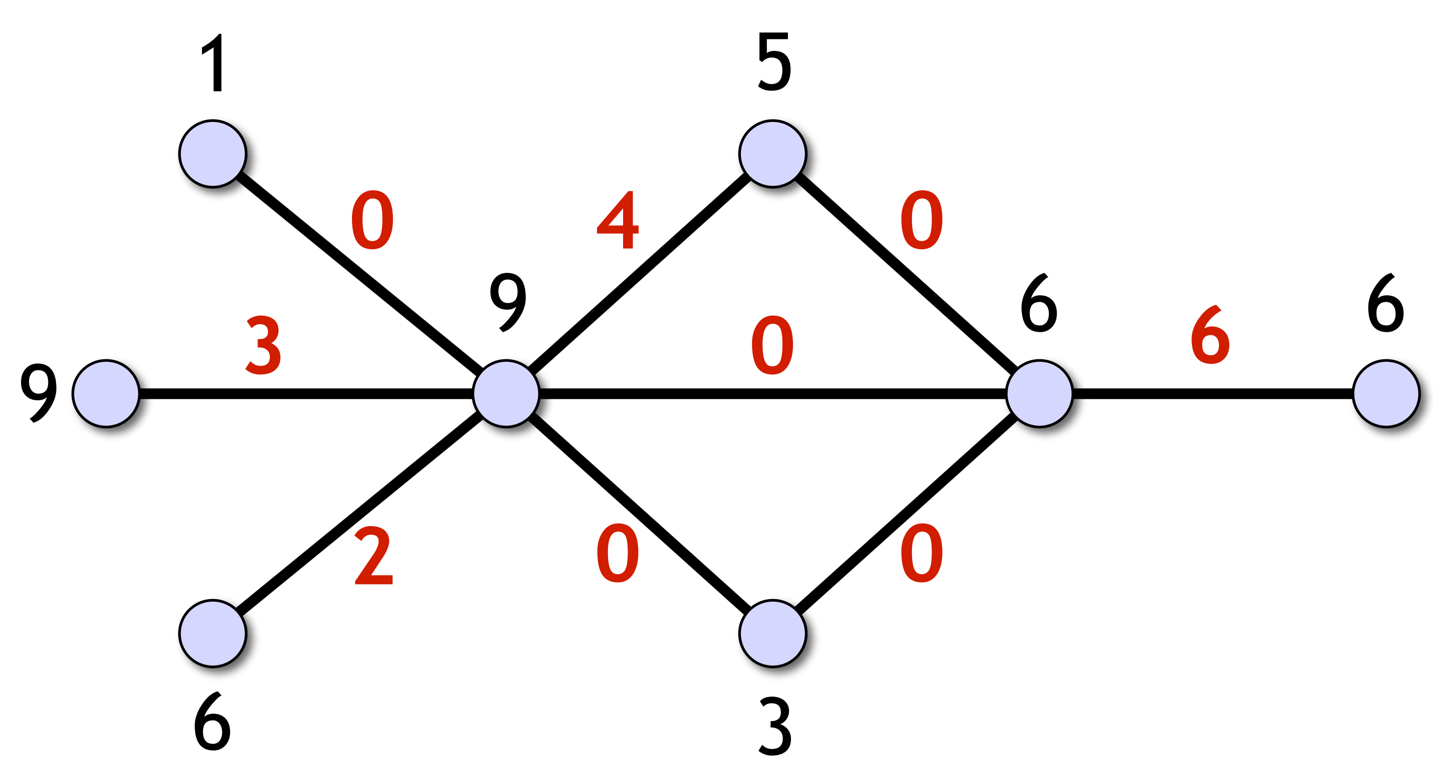
- Convenient to study a more general problem: minimum-weight vertex cover
  - More general problems are sometimes easier to solve?



Notation:
$w(v)$ = weight of v

Edge packings and vertex covers

- **Edge packing**: weight $y(e) \geq 0$ for each edge e
  - Packing constraint: $y[v] \leq w(v)$ for each node v, where $y[v]$ = total weight of edges incident to v



Edge packings and vertex covers

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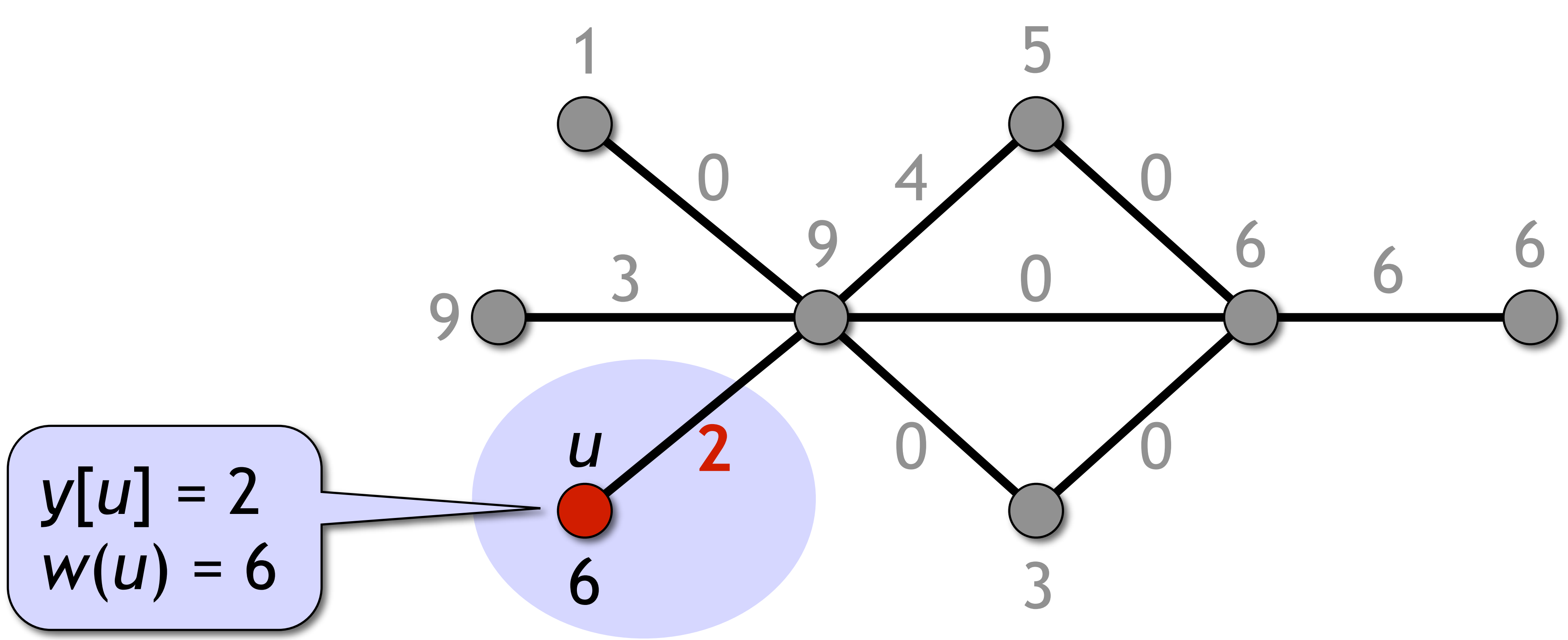

Edge packings and vertex covers

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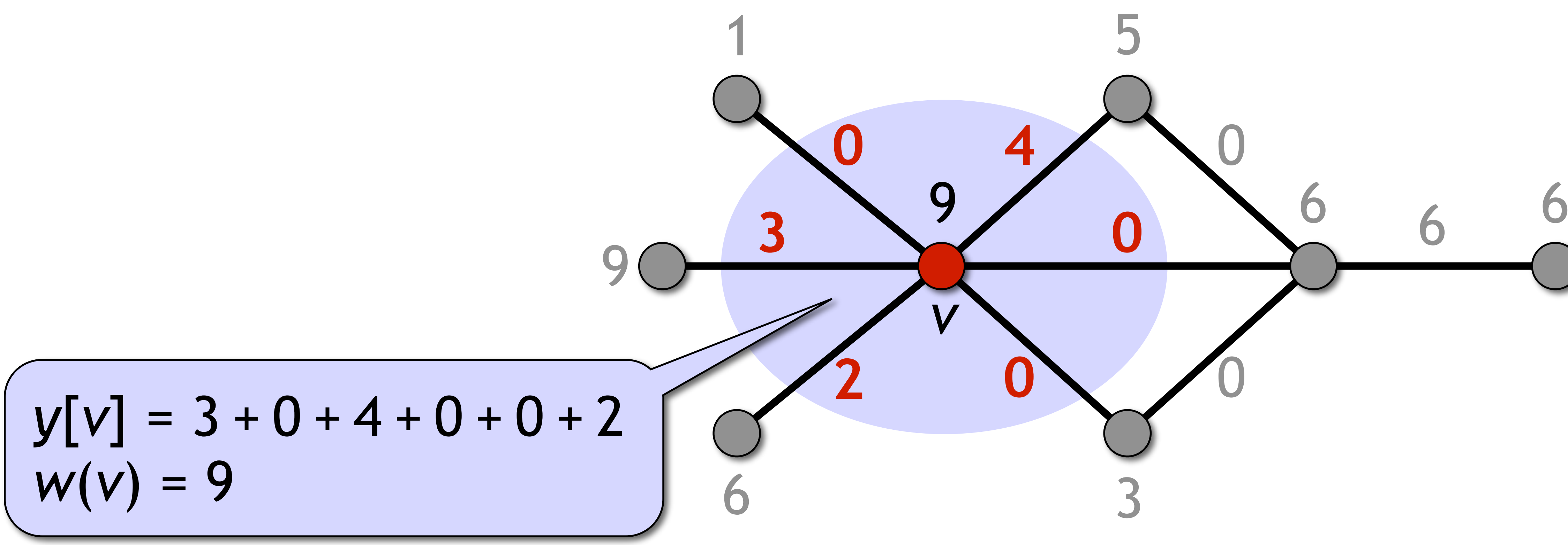
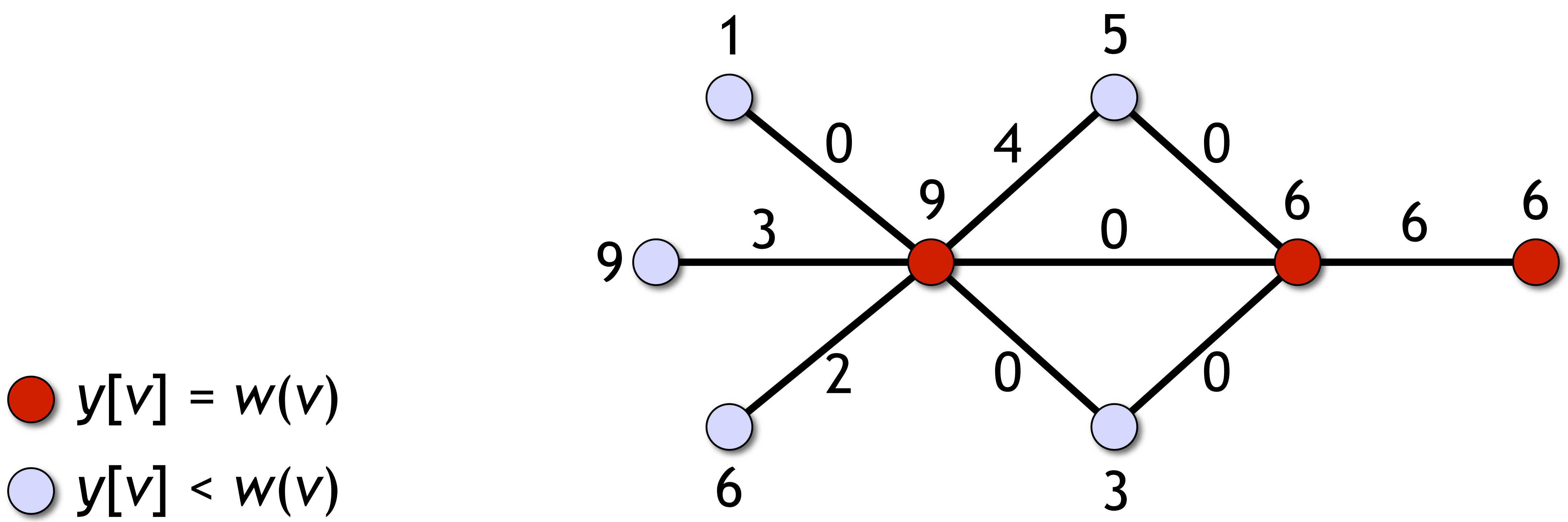


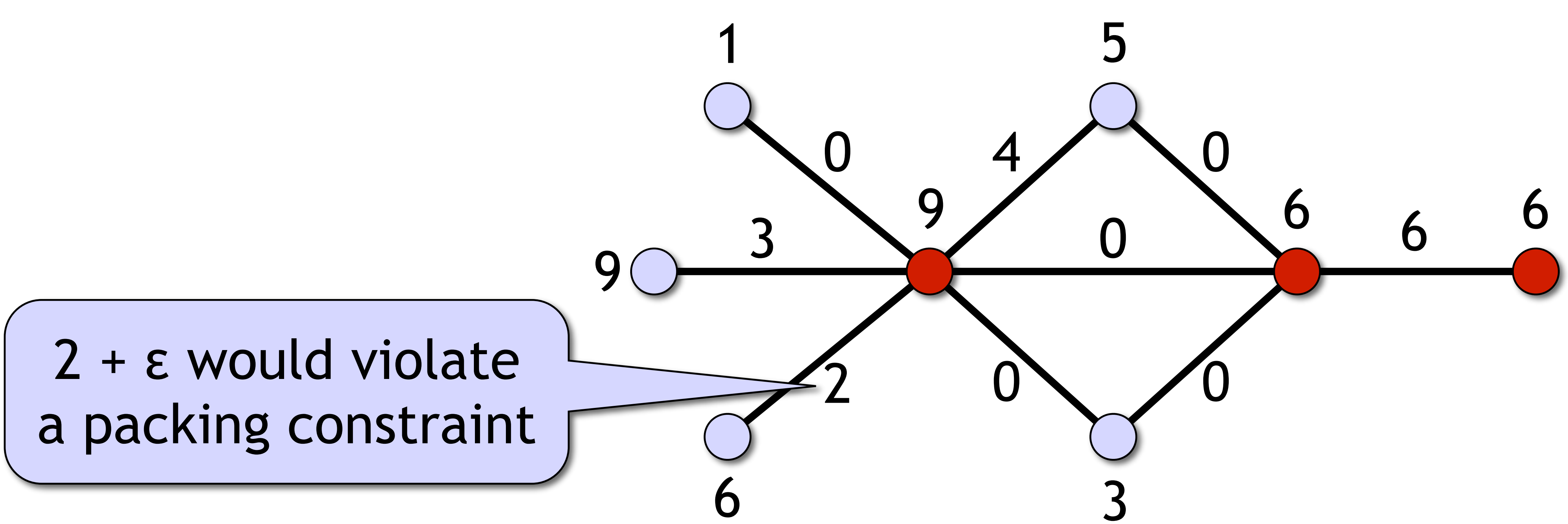
Edge packings and vertex covers

- Node v is **saturated** if $y[v] = w(v)$
  - Total weight of edges incident to v is ***equal*** to $w(v)$, i.e., the packing constraint holds with equality



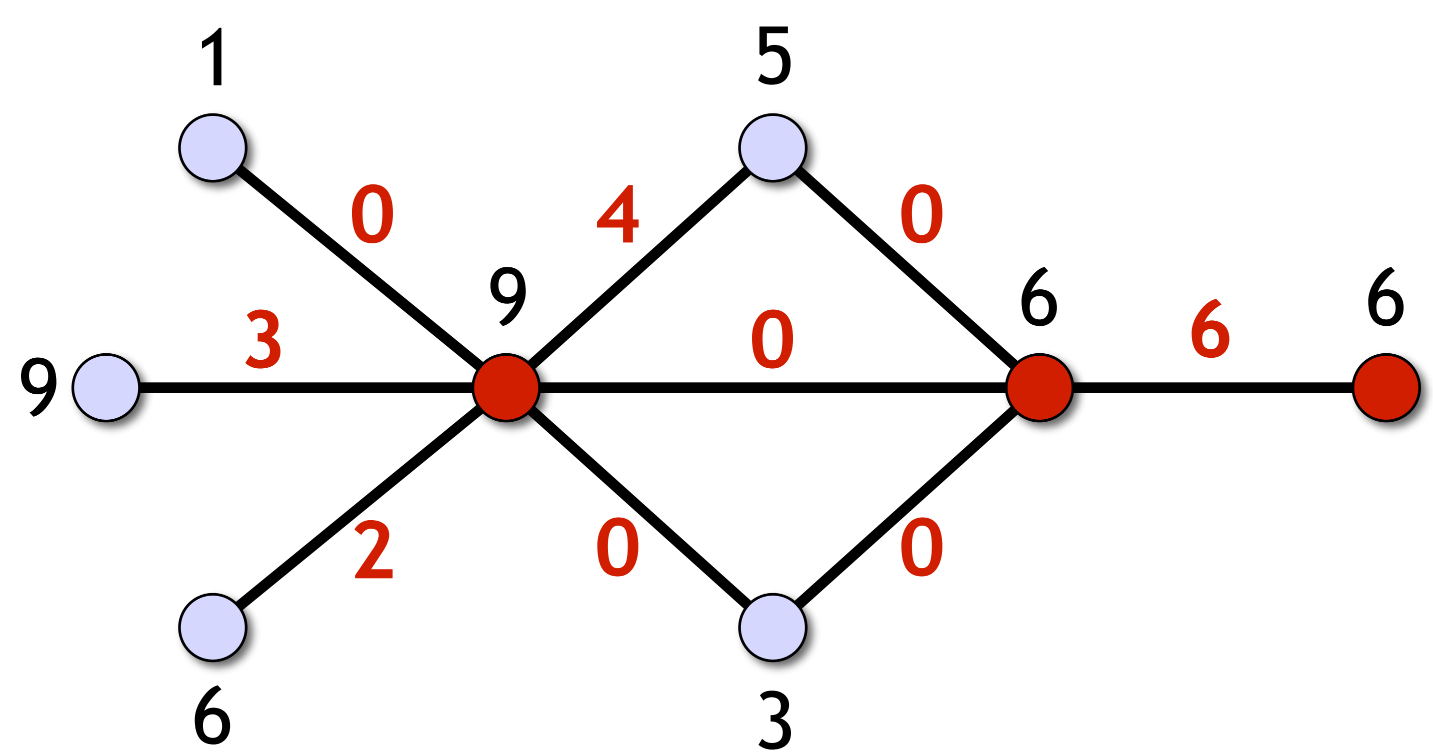
Edge packings and vertex covers

- Edge *e* is **saturated** if at least one endpoint of *e* is saturated
  - Equivalently: edge weight *y*(*e*) can't be increased



Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  $\Longleftrightarrow$ none of the edge weights $y(e)$ can be increased
  $\Longleftrightarrow$ saturated nodes form a vertex cover!



Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find:
  - Centralised setting: NP-hard
  - Distributed setting: integer problem (choose 0 or 1), symmetry-breaking issues
- Maximal edge packing y easy to find:
  - Centralised setting: trivial greedy algorithm
  - Distributed setting: linear problem, no symmetry-breaking issues (?)

Edge packings and vertex covers

- Minimum-weight vertex cover C^* difficult to find
- Maximal edge packing y easy to find?
- Saturated nodes $C(y)$ in y: **2-approximation** of C^*
  - Textbook proof: LP-duality, relaxed complementary slackness
  - Minimum-weight fractional vertex cover and maximum-weight edge packing are ***dual problems***
  - But we there's a simple and more elementary proof...

Edge packings and vertex covers

	$\sum_{v \in C(y)} w(v)$	Total weight of saturated nodes
$=$	$\sum_{v \in C(y)} y[v]$	Saturated nodes have $y[v] = w(v)$
$=$	$\sum_{e \in E} y(e)\ \lvert e \cap C(y) \rvert$	Interchange the order of summation
$\leq$	$2 \sum_{e \in E} y(e)\ \lvert e \cap C^* \rvert$	Each edge is covered at least ***once*** by C^* and at most ***twice*** by $C(y)$
$=$	$2 \sum_{v \in C^*} y[v]$	Interchange the order of summation
$\leq$	$2 \sum_{v \in C^*} w(v)$	All nodes have $y[v] \leq w(v)$

Edge packings and vertex covers

$\sum_{v\in C(y)} \sum_{e\in E:\ v\in e} y(e)$

$\sum_{v\in C(y)} w(v)$ ted nodes

$= \sum_{v\in C(y)} y[v]$ $y[v] = w(v)$

$\sum_{e\in E} \sum_{v\in C(y):\ v\in e} y(e)$

$= \sum_{e\in E} y(e)\ |e \cap C(y)|$ Interchange the order of summation

$\leq 2 \sum_{e\in E} y(e)\ |e \cap C^*|$ Each edge is covered at least ***once*** by C^* and at most ***twice*** by $C(y)$

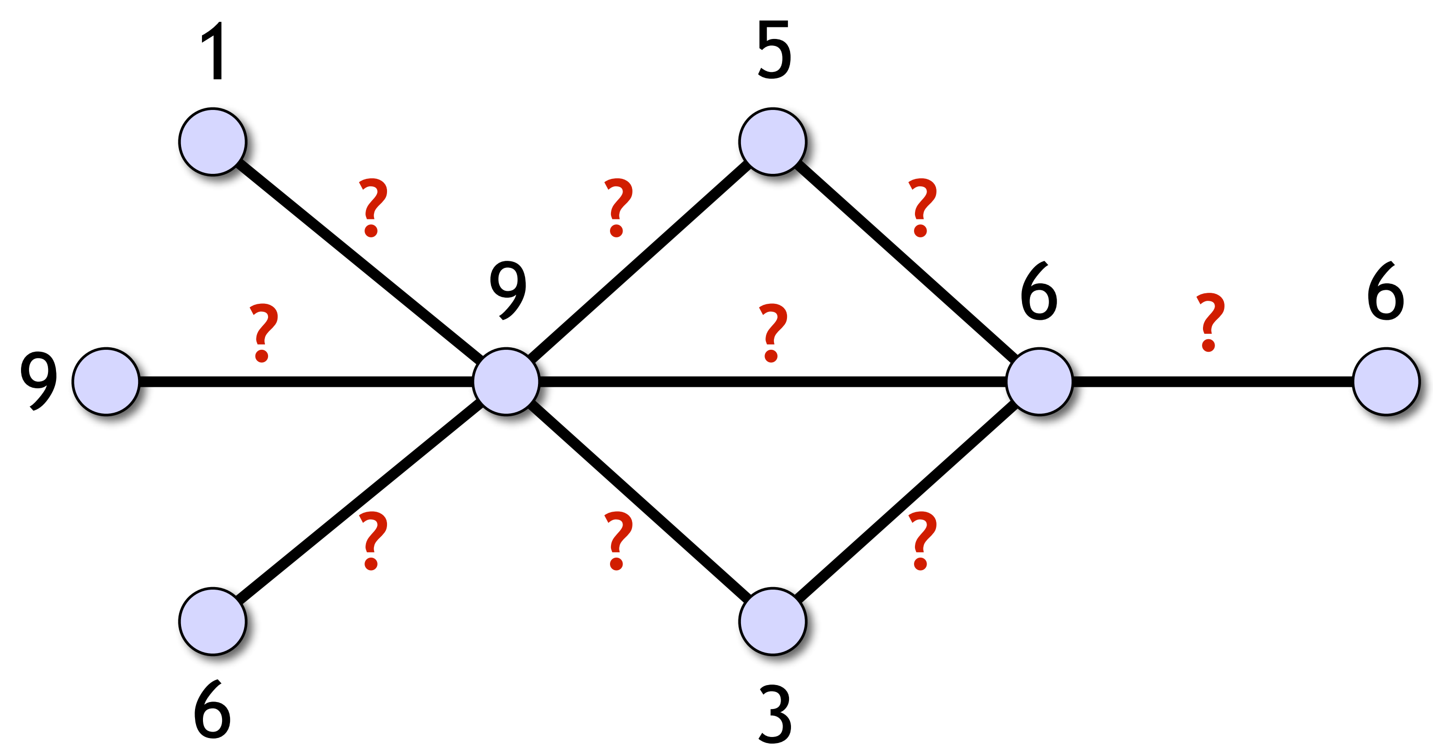
$= 2 \sum_{v\in C^*} y[v]$ Interchange the order of summation

$\leq 2 \sum_{v\in C^*} w(v)$ All nodes have $y[v] \leq w(v)$

Part I: Summary

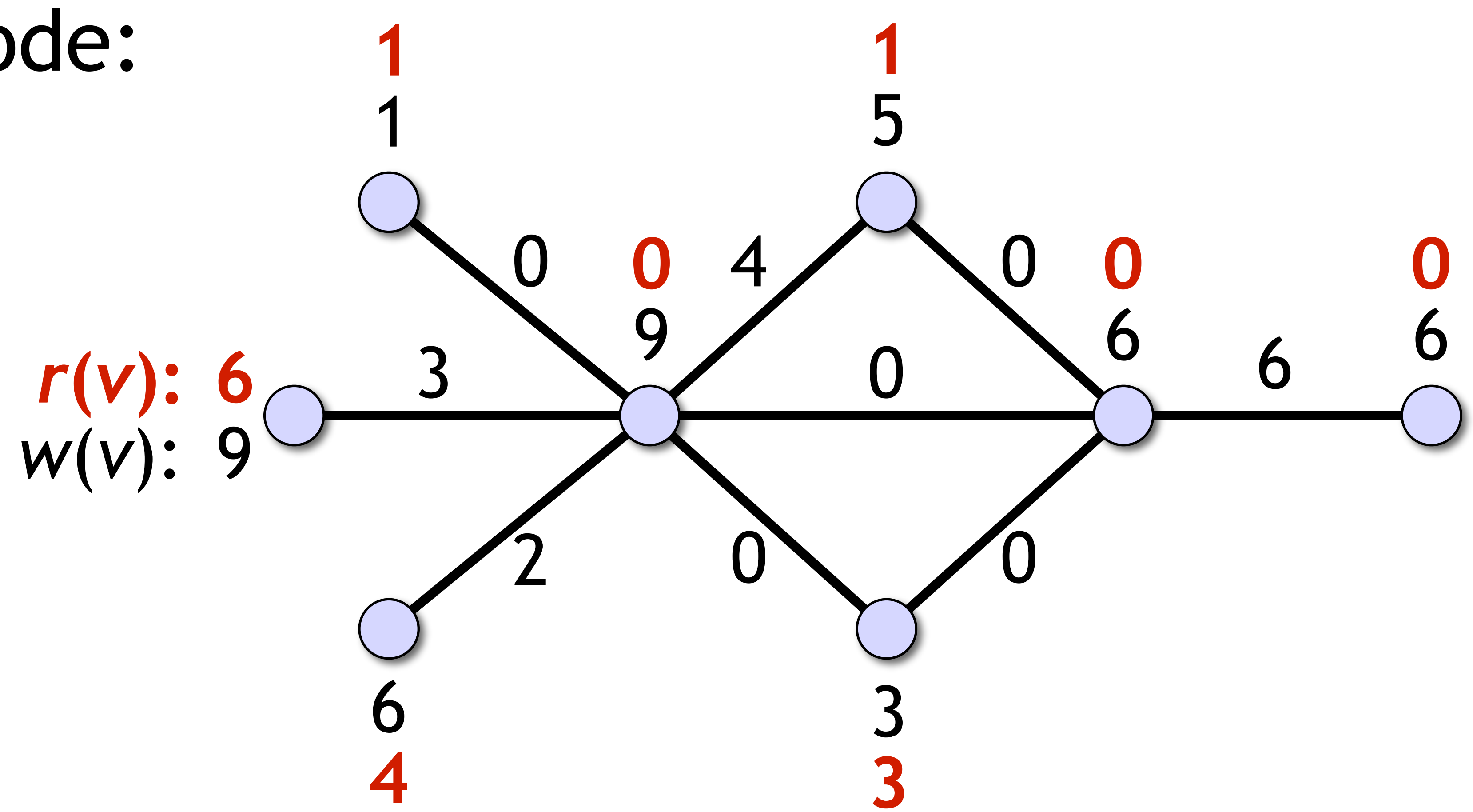
- Goal:
  - Find a 2-approximation of minimum-weight **vertex cover**
  - Deterministic algorithm in the **port-numbering** model
- Idea:
  - Find a **maximal edge packing**, take saturated nodes
- Part II:
  - Begin with a “greedy but safe” algorithm
  - We will see later how the Cole–Vishkin technique helps

Part II:
Finding a maximal edge packing



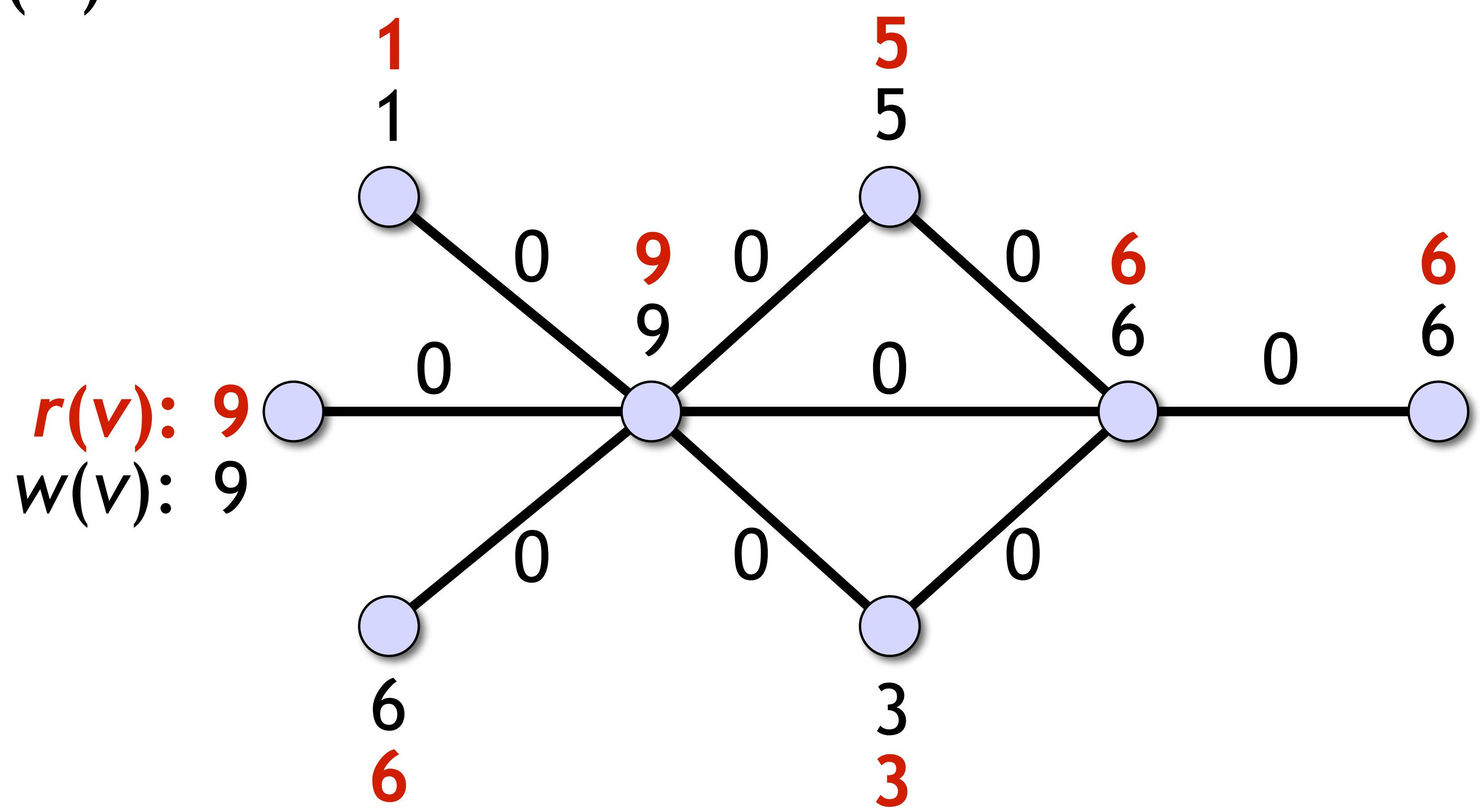
Finding a maximal edge packing: phase I

- $y[v]$ = total weight of edges incident to node v
- **Residual capacity** of node v: $r(v) = w(v) - y[v]$
- Saturated node: $r(v) = 0$



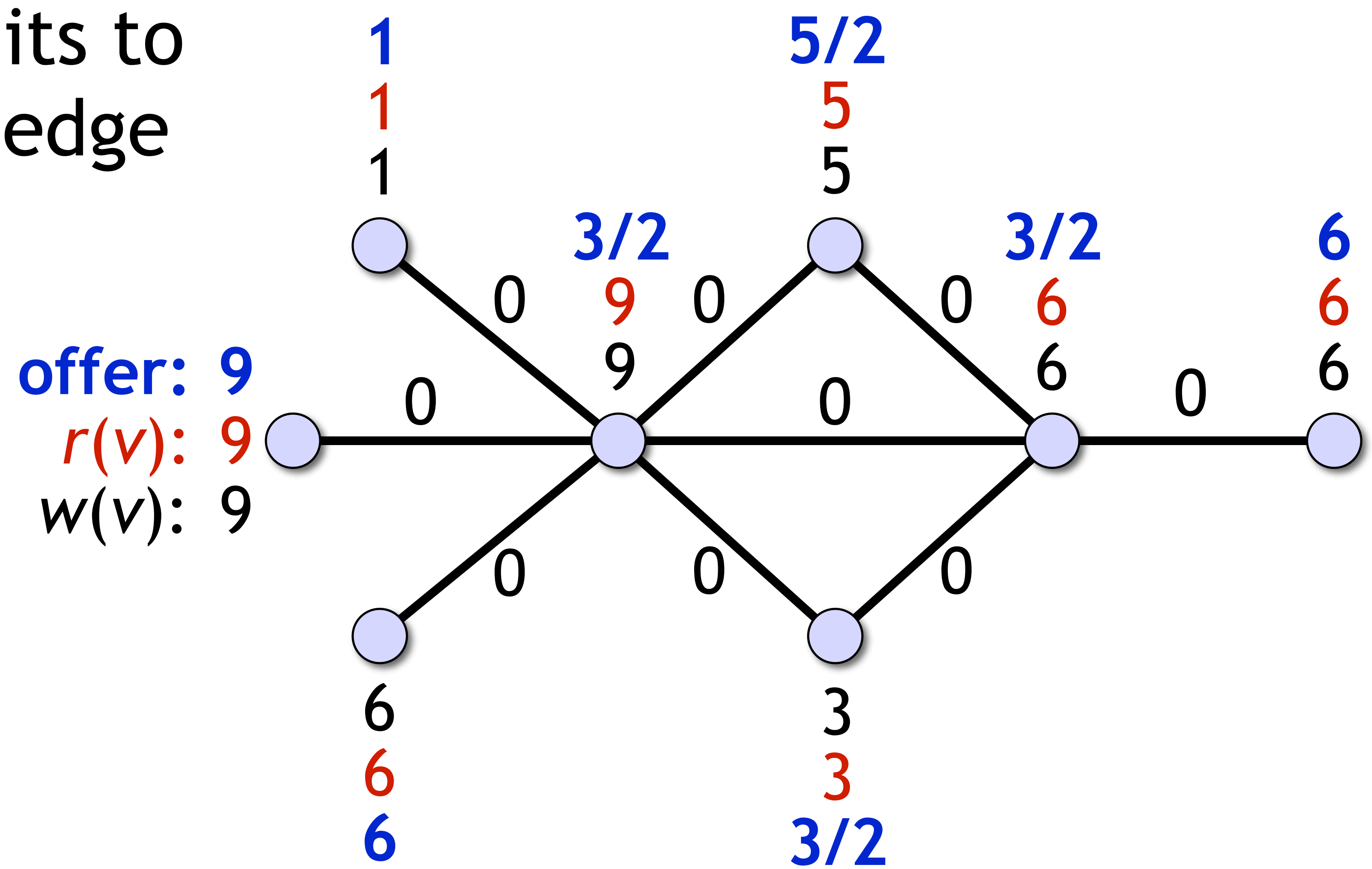
Finding a maximal edge packing: phase I

Start with a trivial edge packing $y(e) = 0$



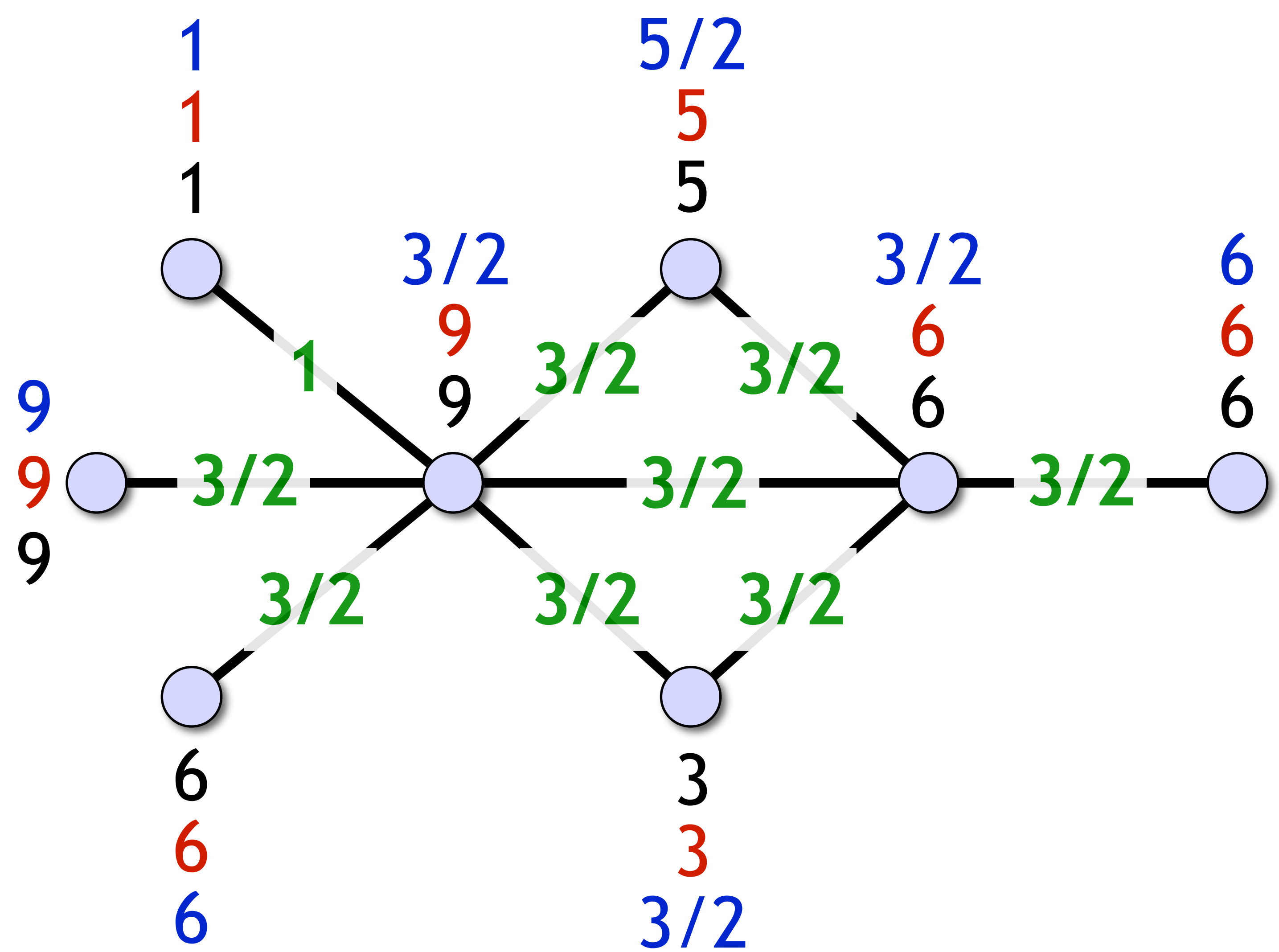
Finding a maximal edge packing: phase I

Each node v **offers** $r(v)/\deg(v)$ units to each incident edge



Finding a maximal edge packing: basic idea

Each edge **accepts** the smallest of the 2 offers it received

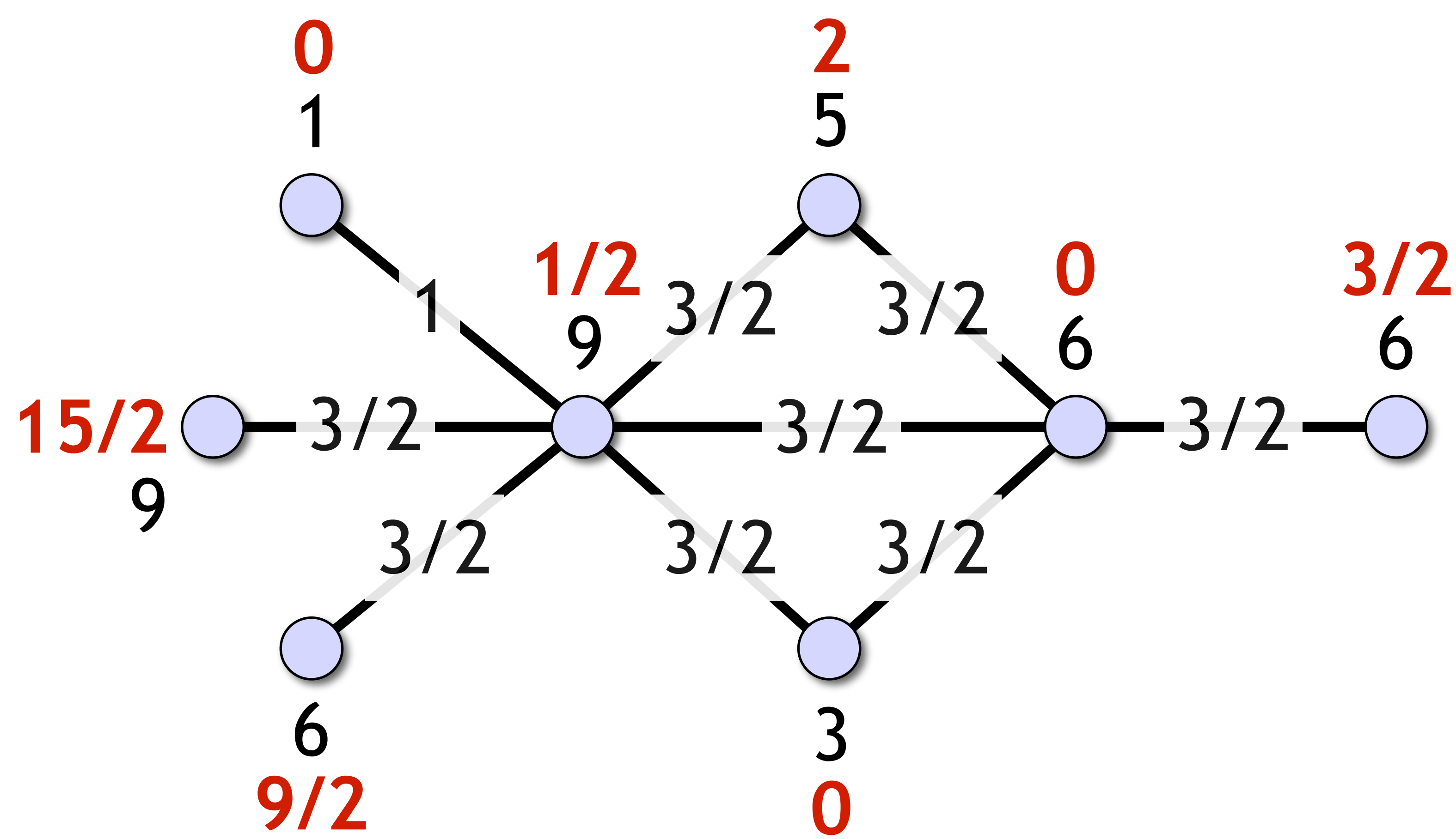


Increase $y(e)$ by this amount

- Safe, can't violate packing constraints

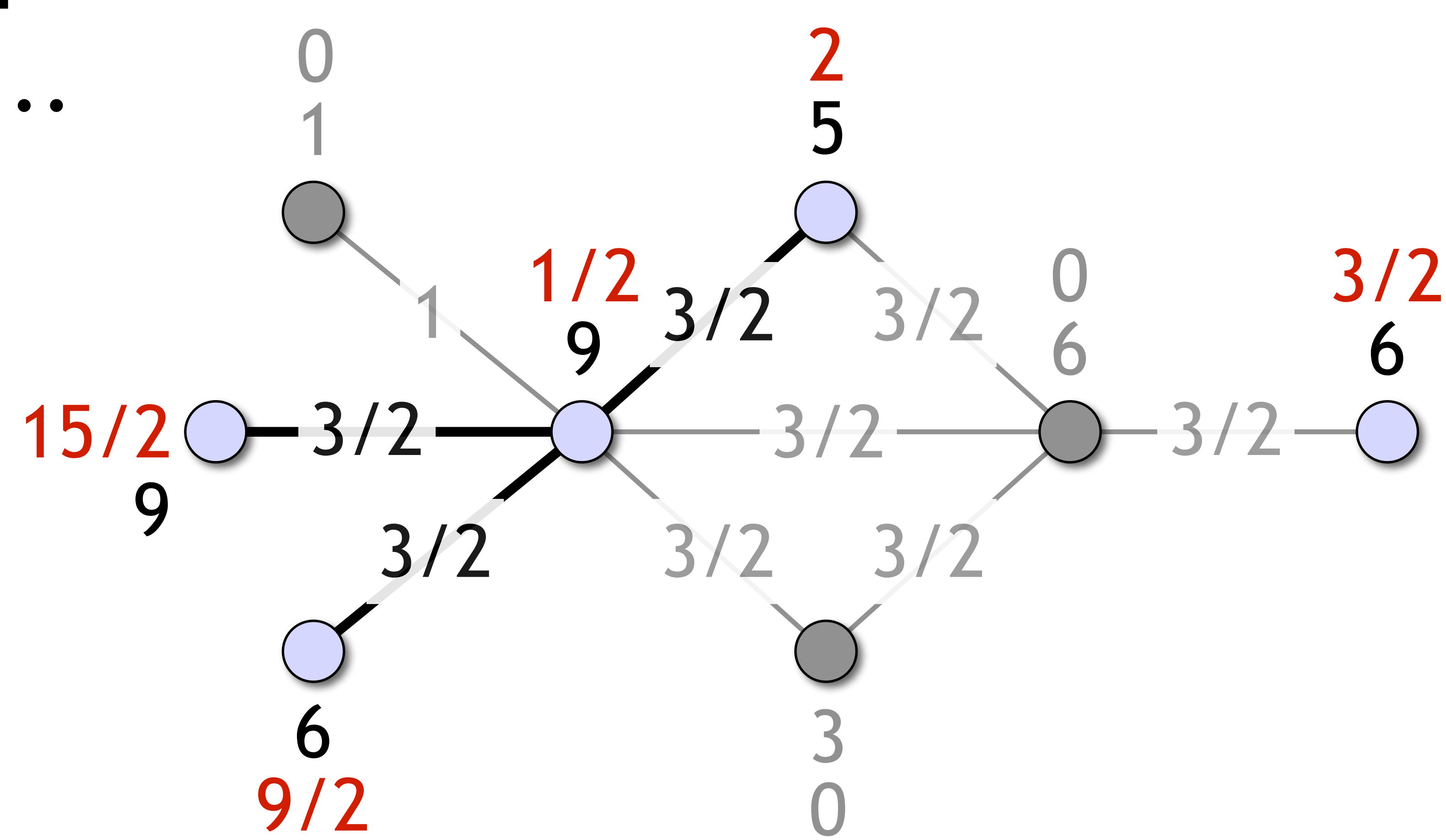
Finding a maximal edge packing: phase I

Update **residuals**...



Finding a maximal edge packing: phase I

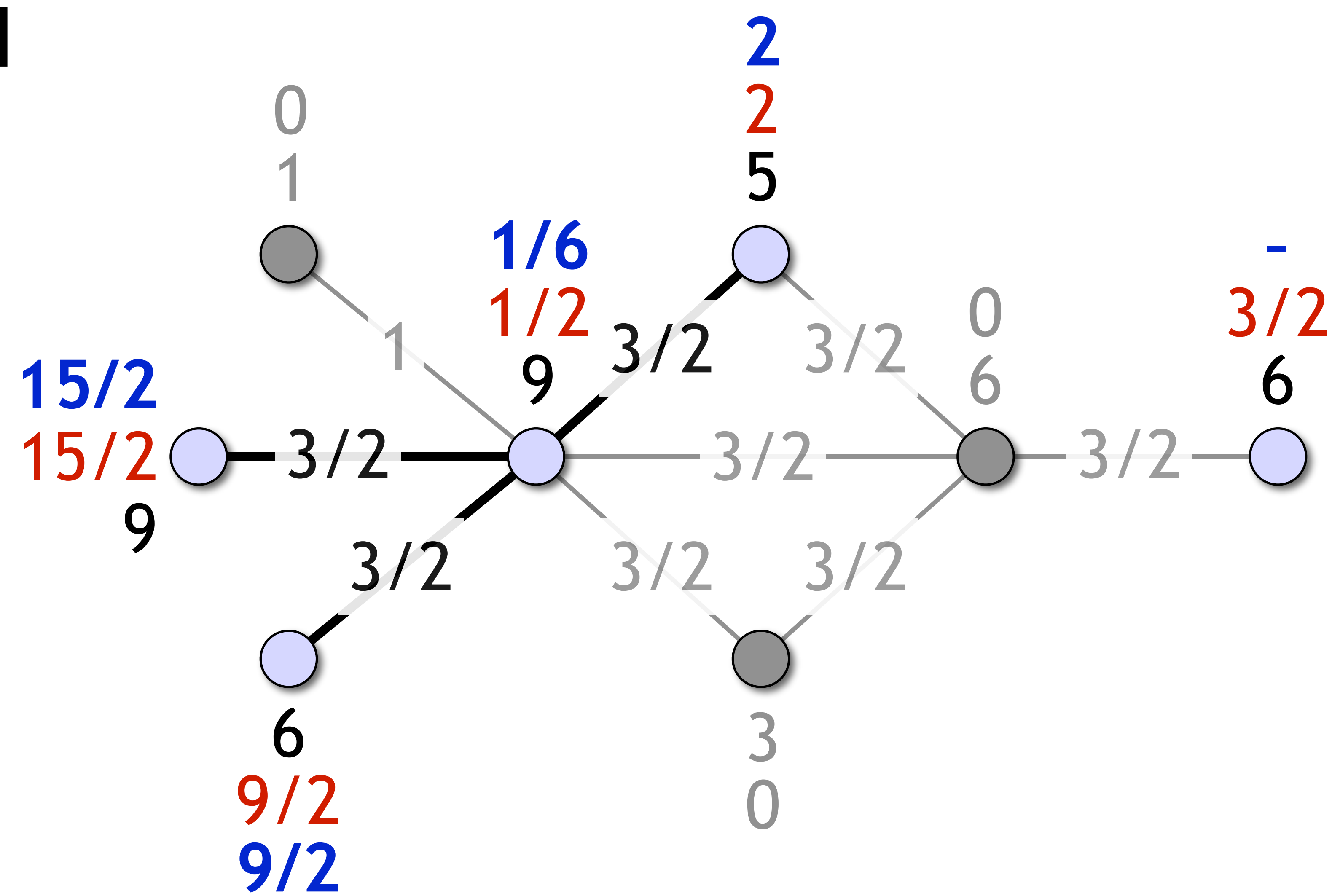
Update residuals, discard saturated nodes and edges...



Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

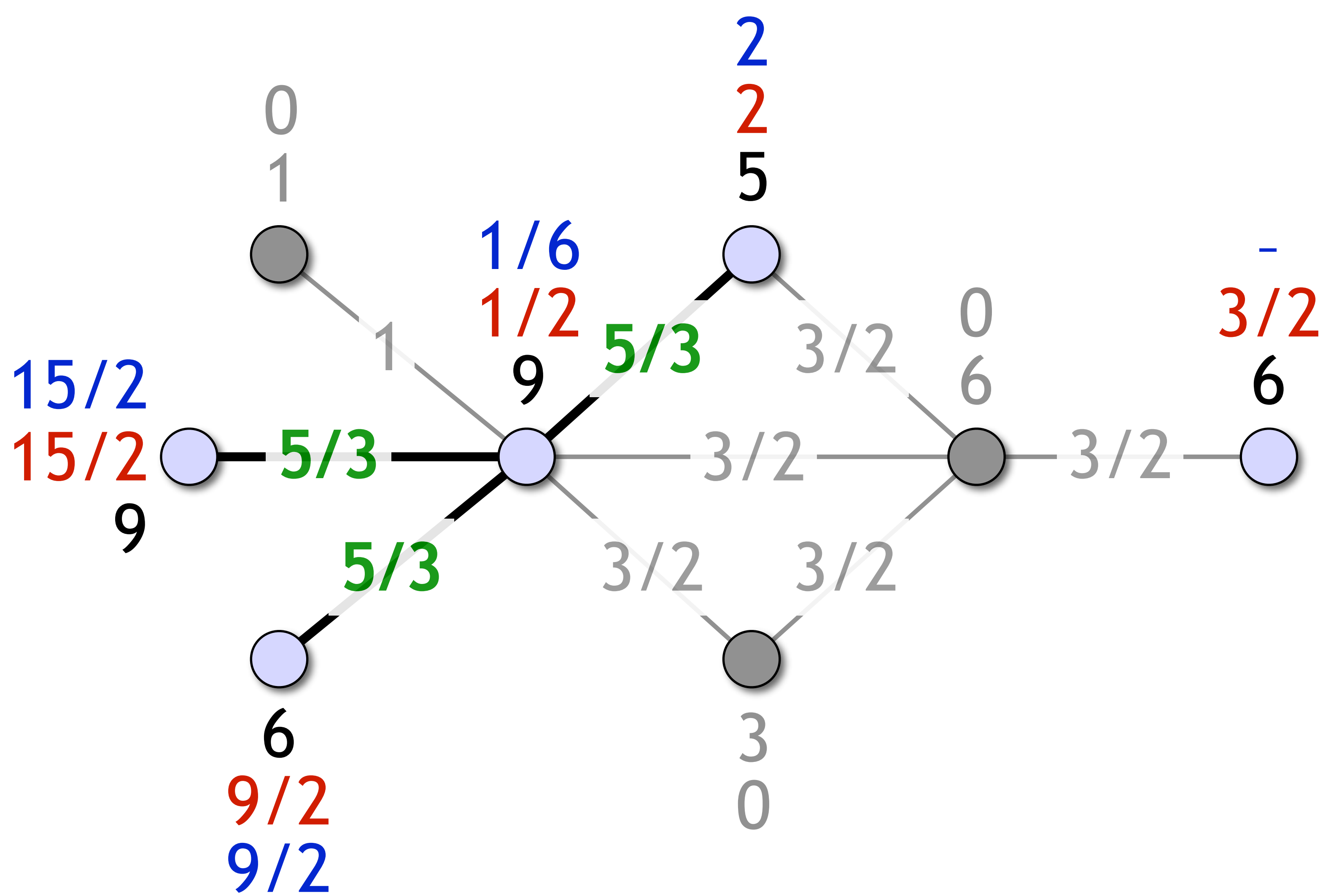


Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...



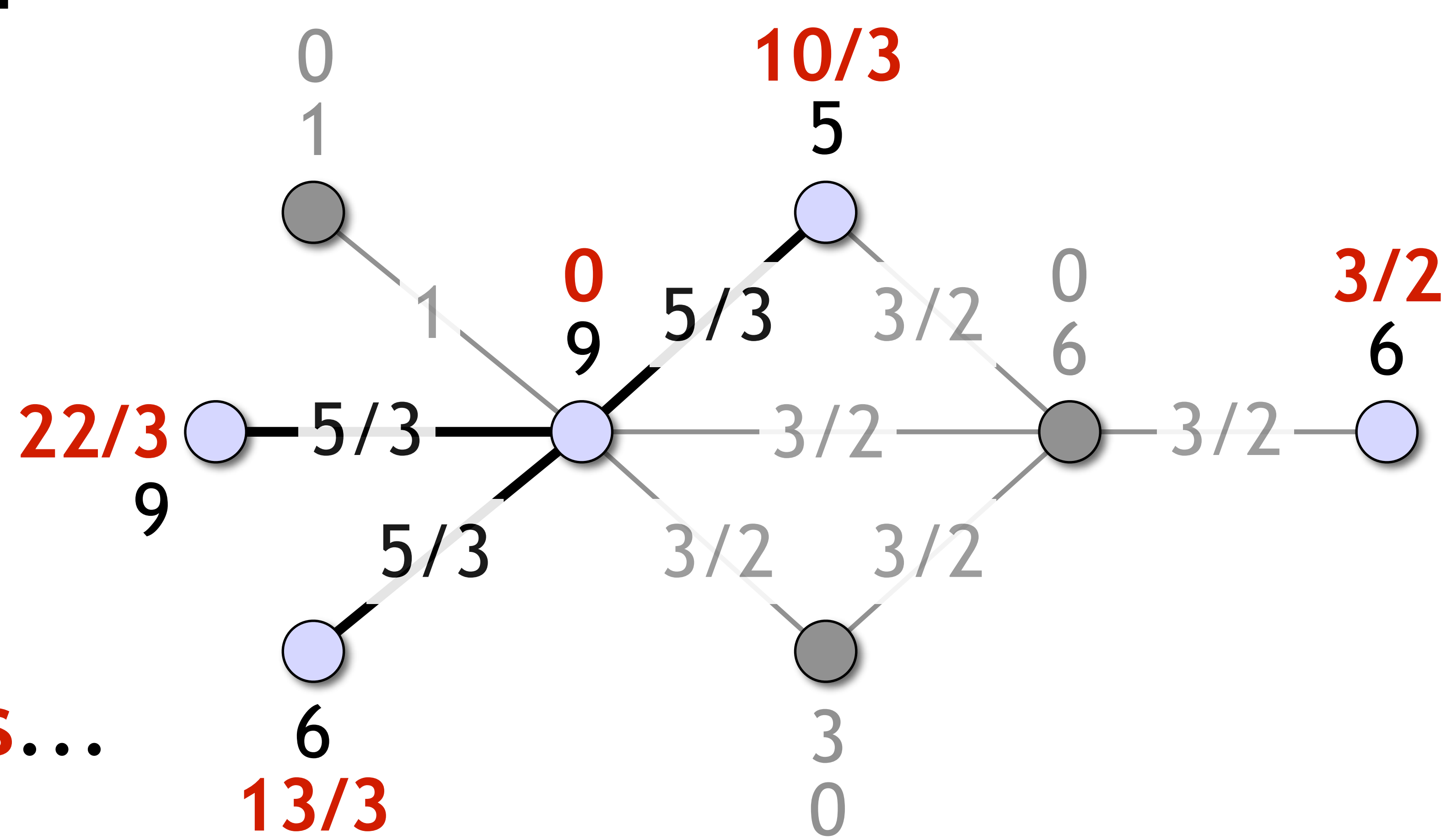
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

Update residuals...



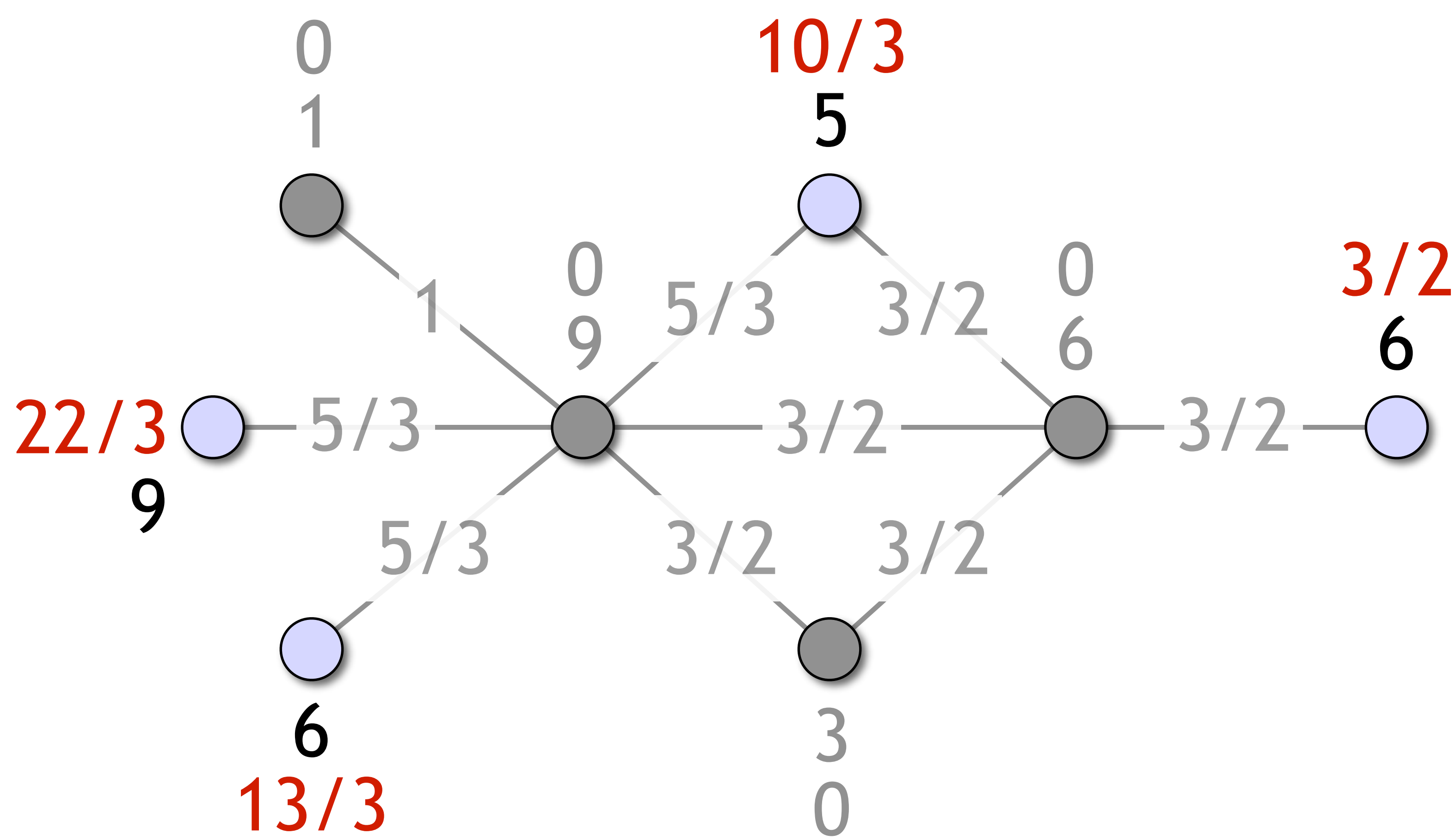
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

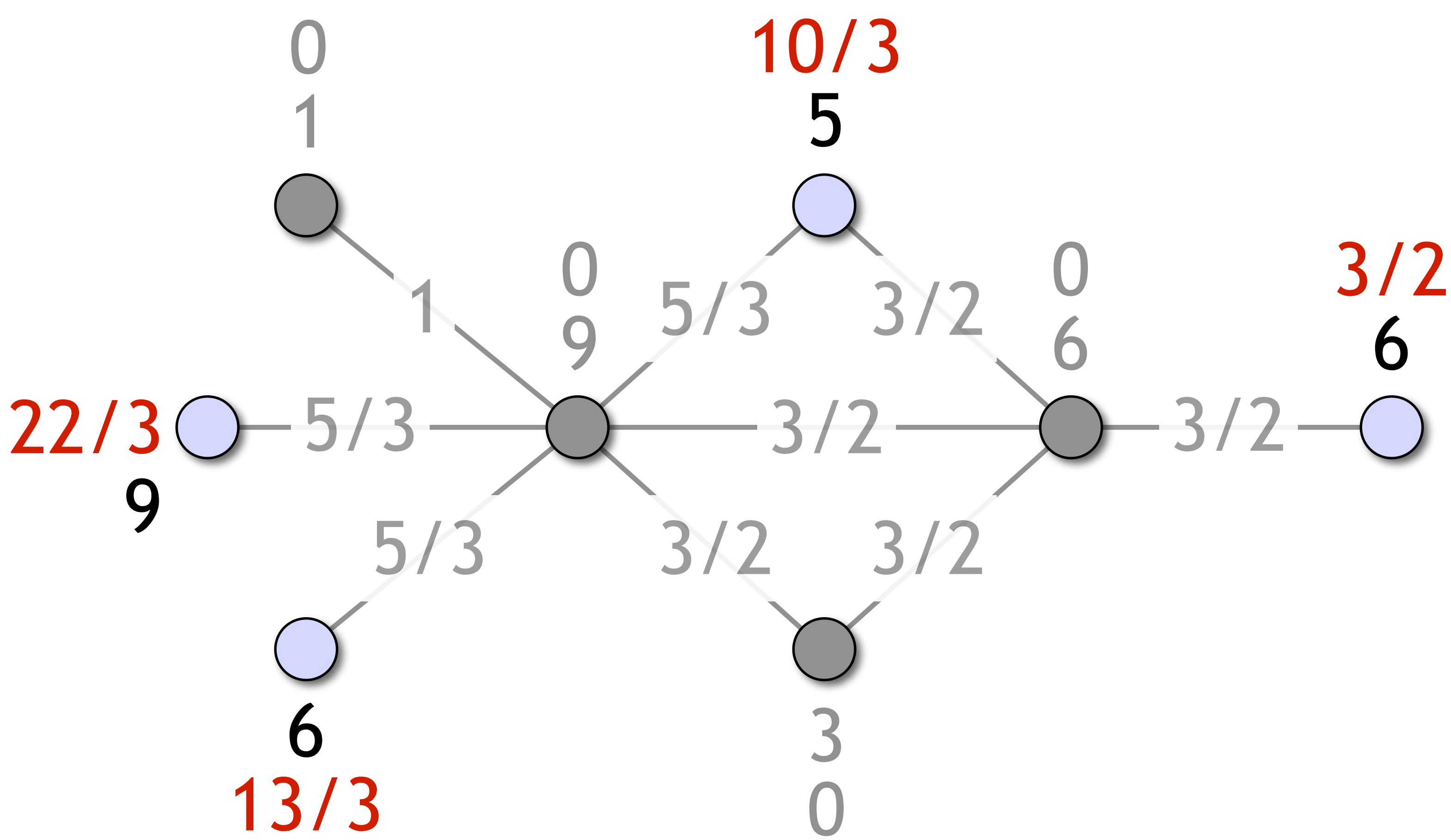
Update residuals and graph, etc.



Finding a maximal edge packing: phase I

This is a simple deterministic distributed algorithm

We are making some progress towards finding a maximal edge packing — but...



Finding a maximal edge packing: phase I

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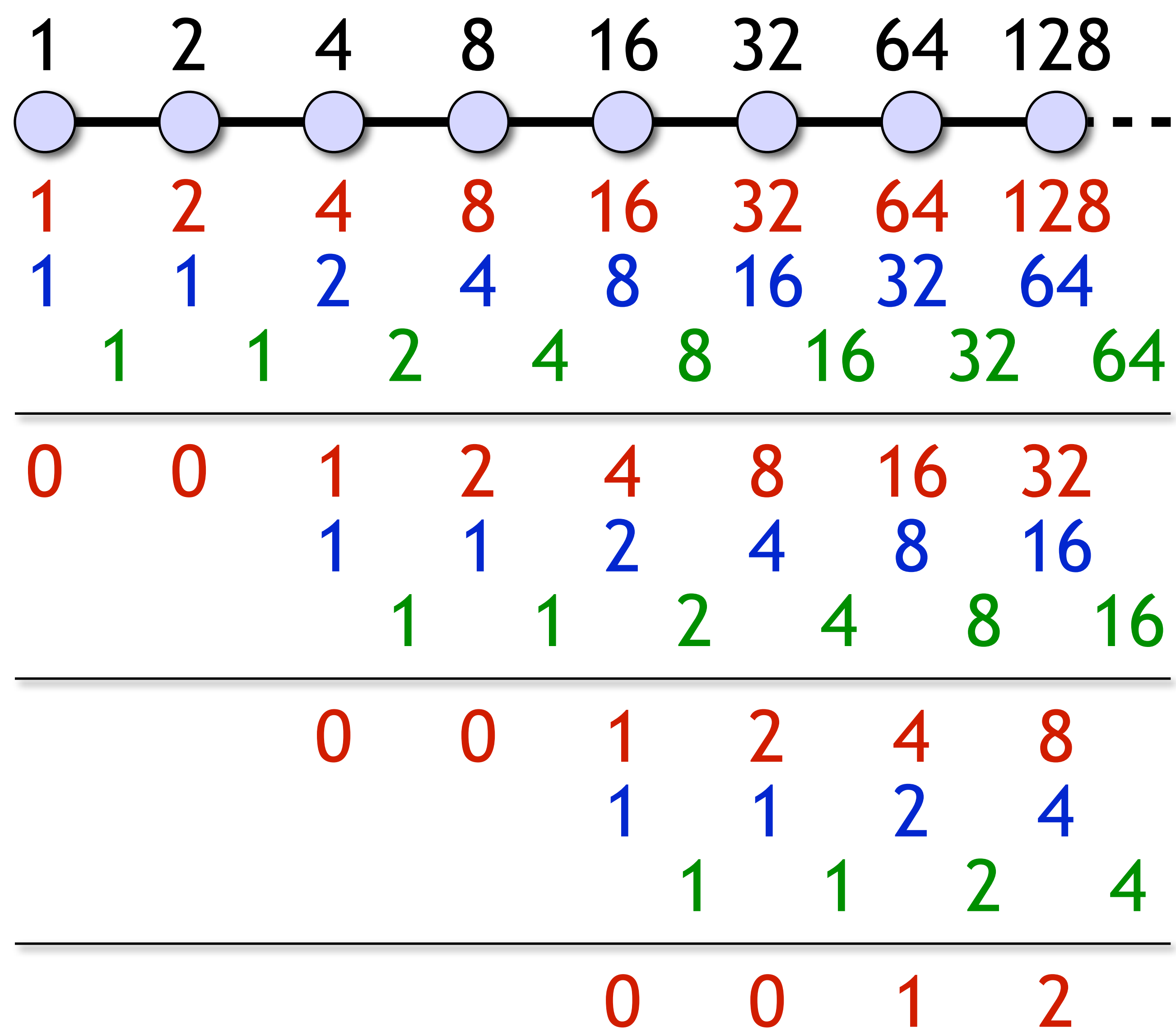
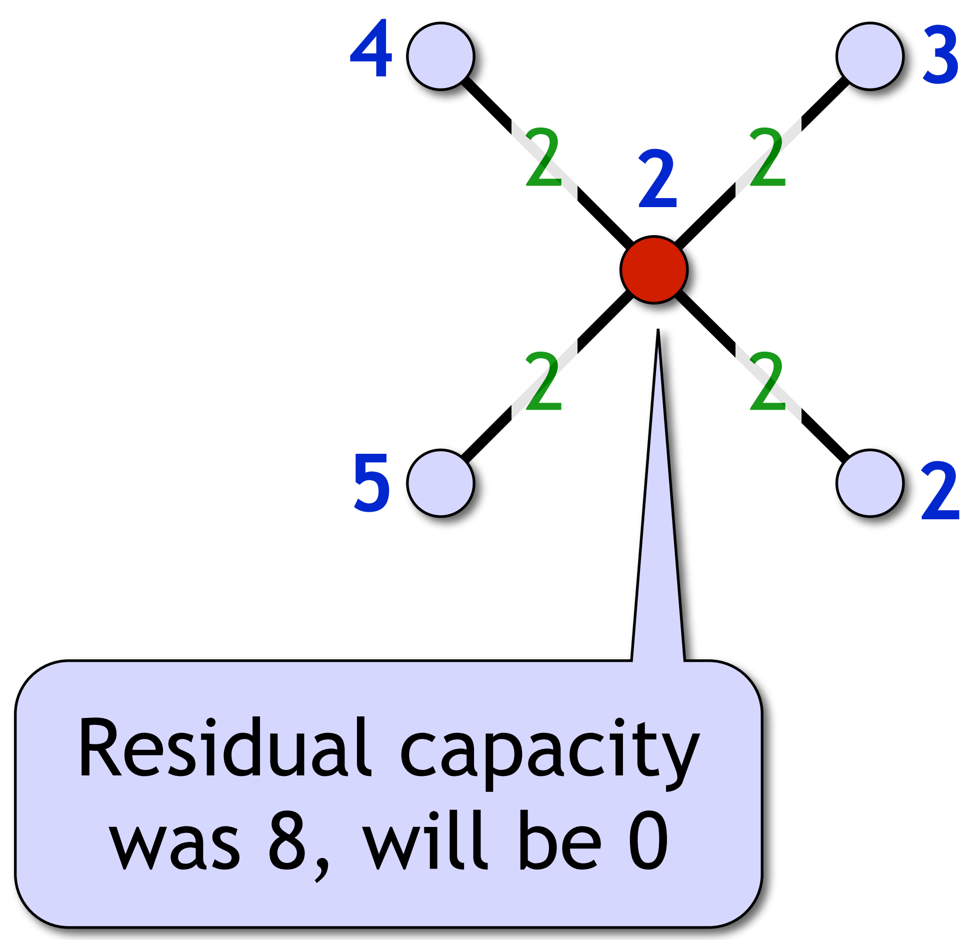


We are making some progress towards finding a maximal edge packing – but this is **too slow**!

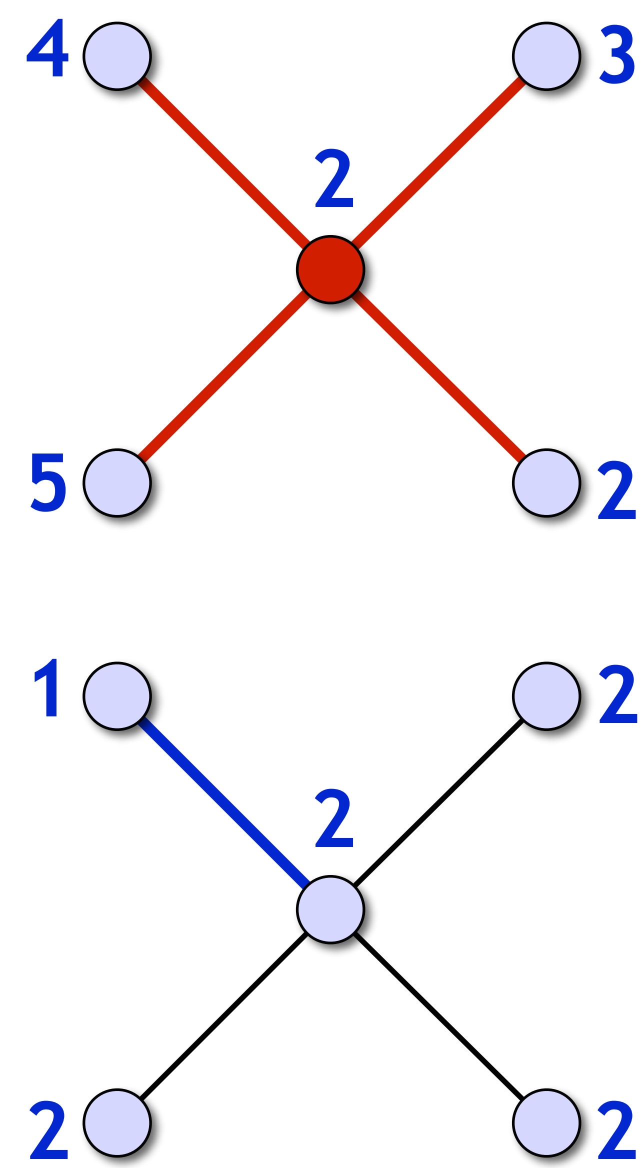
Finding a maximal edge packing: colouring trick

- Offer is a local minimum:
  - Node will be **saturated**
  - And all edges incident to it will be saturated as well



Finding a maximal edge packing: colouring trick

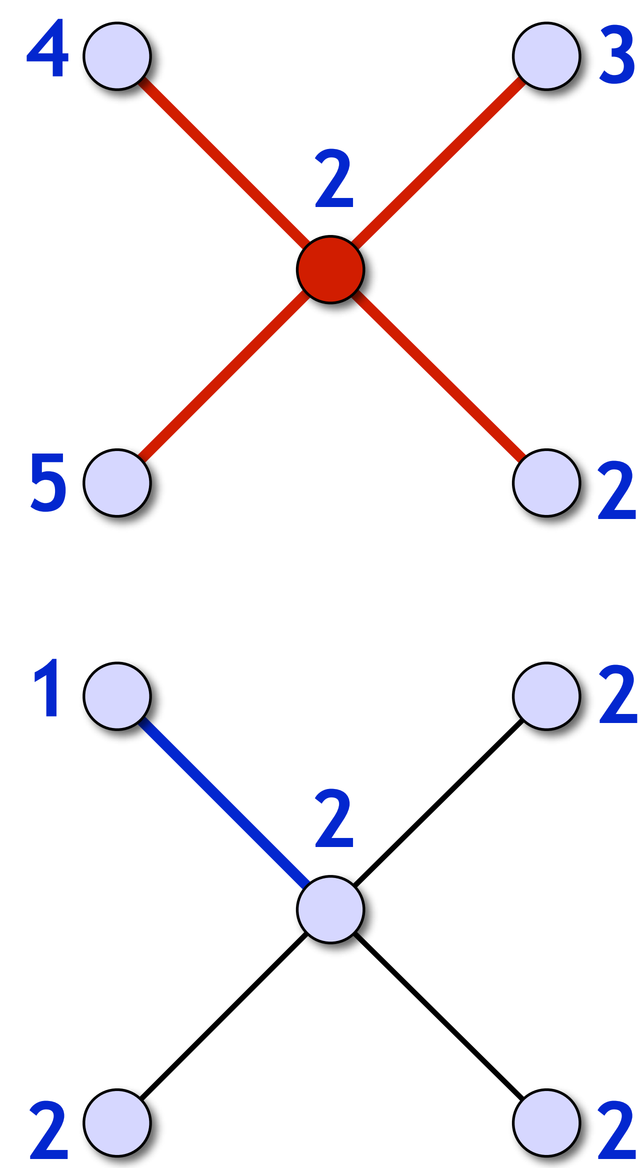
- Offer is a local minimum:
  - Node will be **saturated**
- Otherwise there is a neighbour with a different offer:
  - Interpret the offer sequences as "colours"
  - Nodes *u* and *v* have different colours: $\{u, v\}$ is **multicoloured**



Finding a maximal edge packing: colouring trick

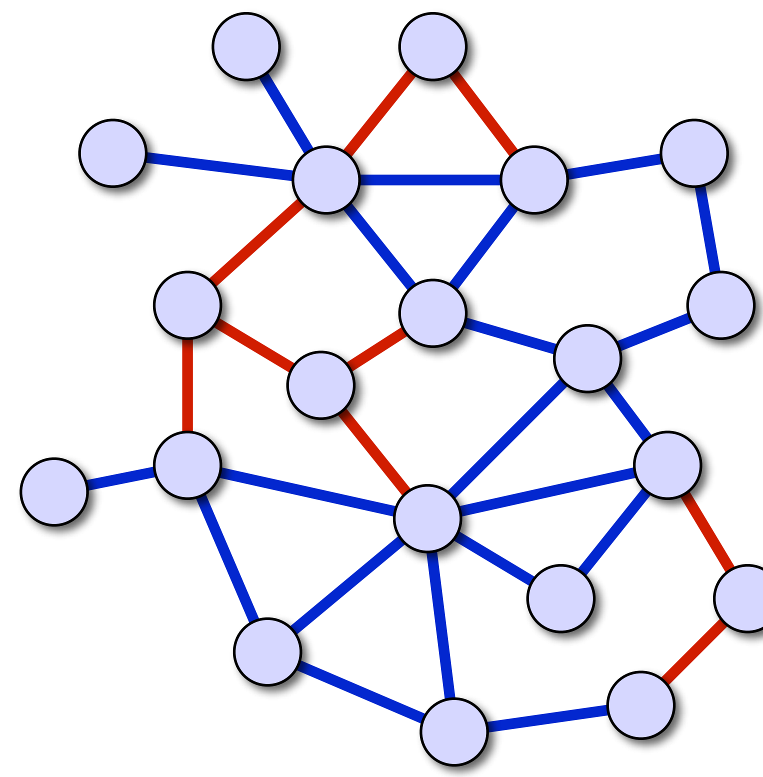
- Some progress guaranteed:
  - On each iteration, for each node, at least one incident edge becomes **saturated** or **multicoloured**
  - Such edges are be discarded in phase I: node degrees decrease by at least one on each iteration
  - Hence in Δ iterations all edges are saturated or multicoloured

Δ = maximum degree



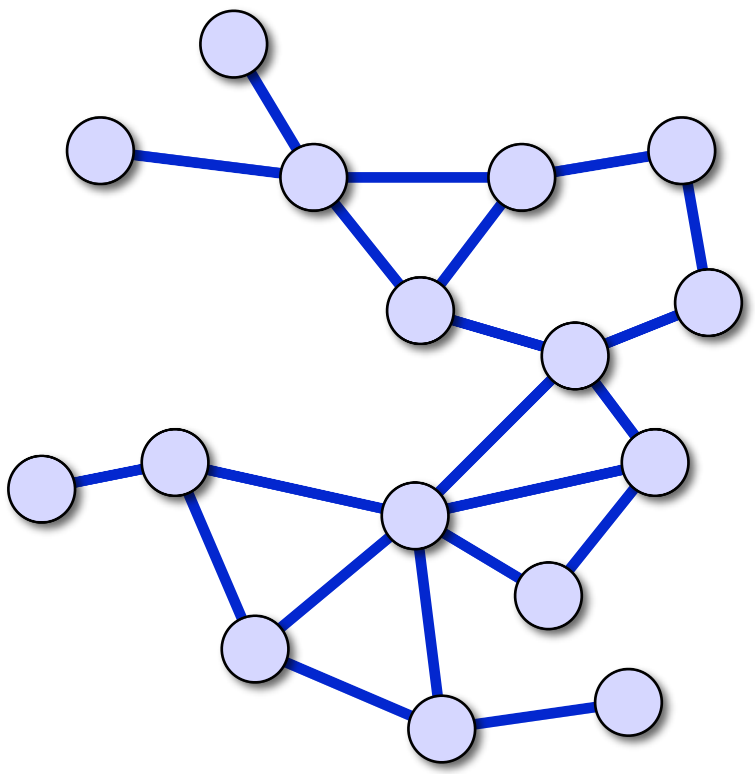
Finding a maximal edge packing: colouring trick

- Phase I: in Δ rounds all edges are **saturated** or **multicoloured**
  - Saturated edges are good — we're trying to construct a maximal edge packing
  - Why are the multicoloured edges useful?



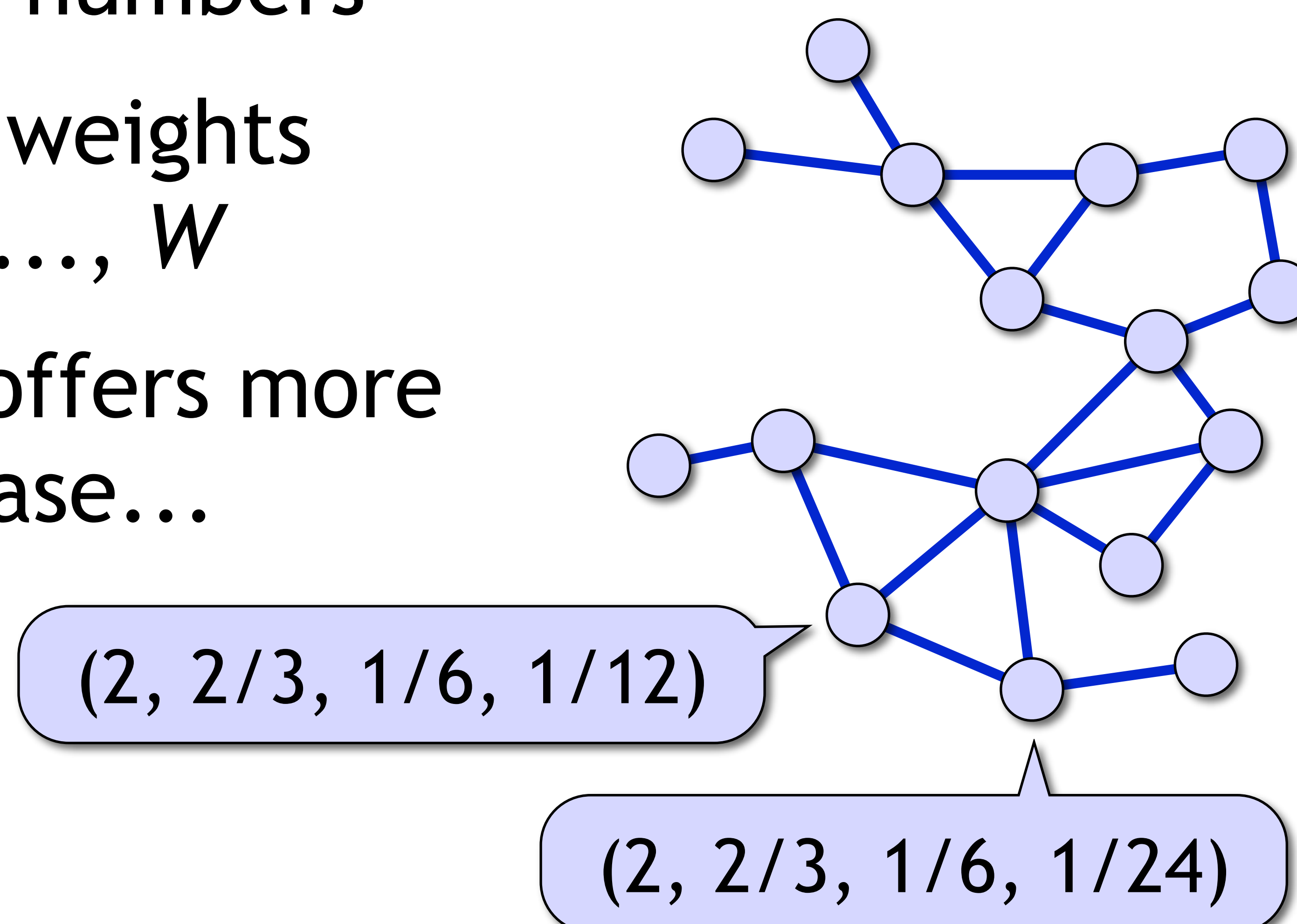
Finding a maximal edge packing: colouring trick

- Phase I: in Δ rounds all edges are **saturated** or **multicoloured**
  - Saturated edges are good — we're trying to construct a maximal edge packing
  - Why are the multicoloured edges useful?
  - Let's focus on unsaturated nodes and edges



Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ offers, which are rational numbers
- Assume that node weights are integers 1, 2, ..., *W*
- Let's analyse the offers more carefully in that case...



Finding a maximal edge packing: colouring trick

- Offers are rationals of the form $q/(\Delta!)^\Delta$
  - Proof idea: multiply weights by $(\Delta!)^\Delta$
  - Then $r(v)$ is a multiple of $(\Delta!)^\Delta$ before iteration 1
  - Offer $r(v)/\deg(v)$ is a multiple of $(\Delta!)^{\Delta-1}$ on iteration 1
  - $r(v)$ is a multiple of $(\Delta!)^{\Delta-1}$ after iteration 1

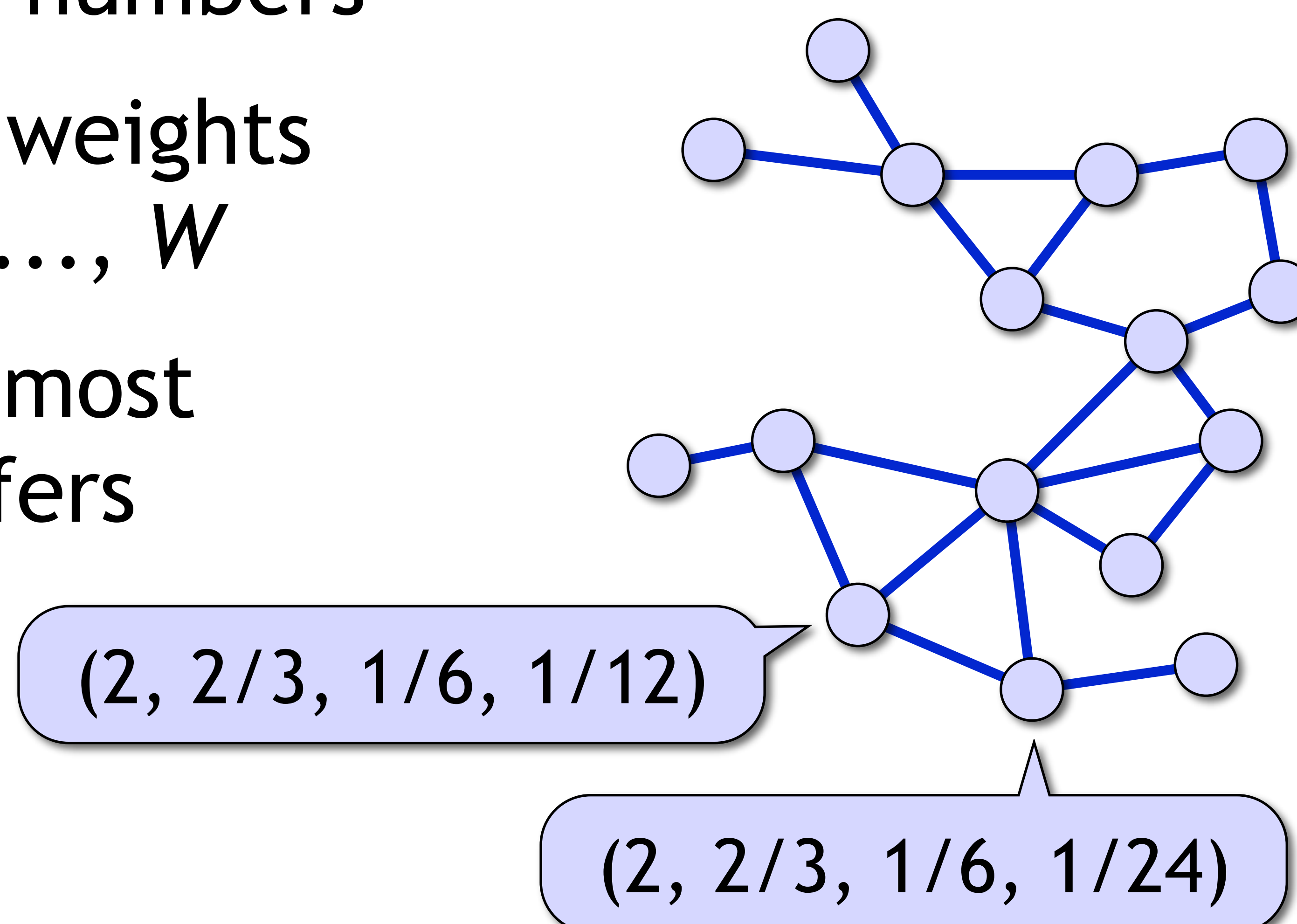
    ... (more formally: proof by induction)
  - $r(v)$ is a multiple of $\Delta!$ before iteration Δ
  - Offers are integers on iteration Δ

Finding a maximal edge packing: colouring trick

- Offers are rationals of the form $q/(\Delta!)^{\Delta}$
  - Proof idea: if we multiplied weights by $(\Delta!)^{\Delta}$, then the offers would integers throughout the algorithm
  - Without scaling, we get in the worst case $q/(\Delta!)^{\Delta}$
- If node weights are integers 1, 2, ..., W, then offers are rationals between 0 and W
  - Offer of v is at most $r(v) \leq w(v) \leq W$
- There are at most $W(\Delta!)^{\Delta}$ possible offers!

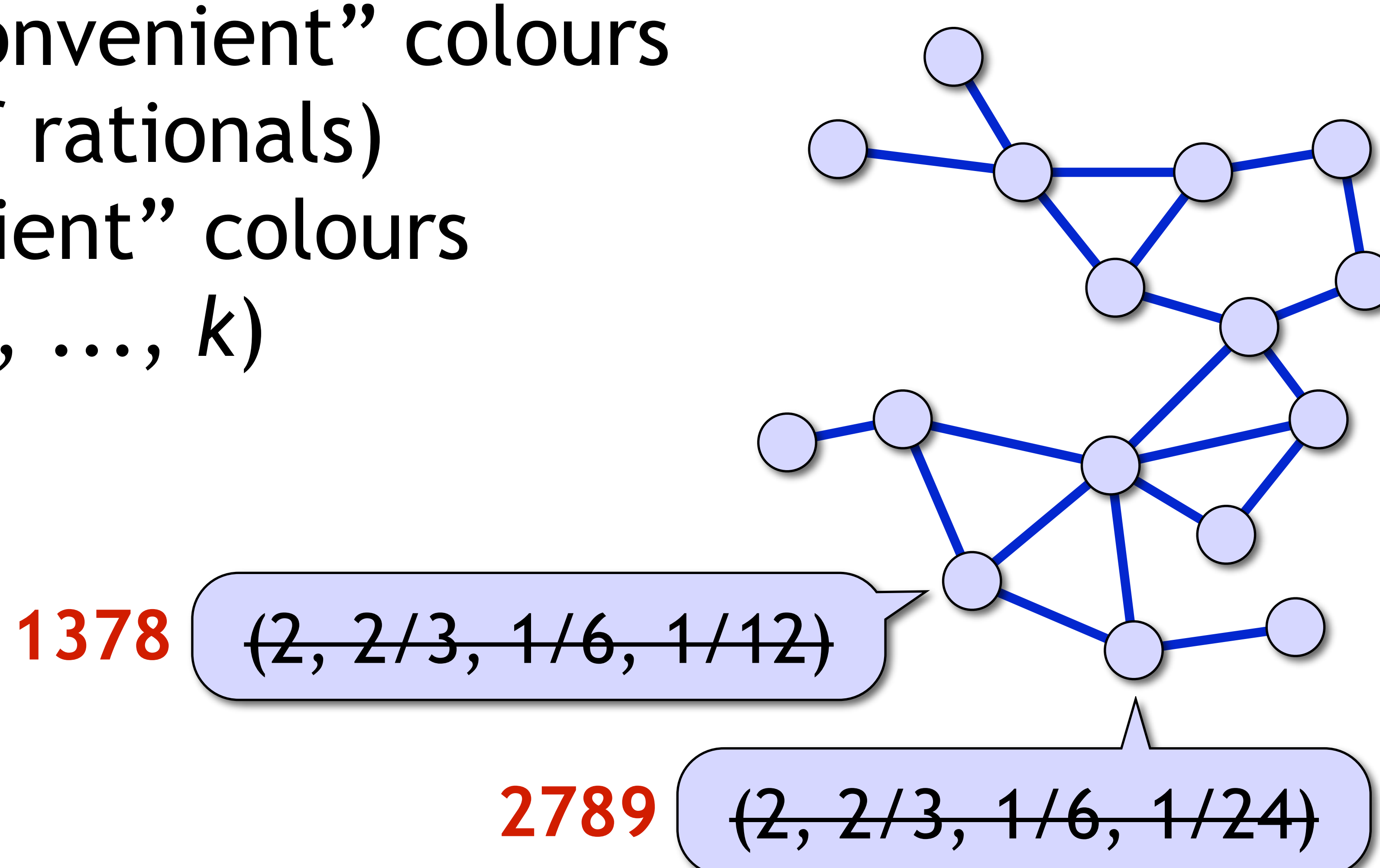
Finding a maximal edge packing: colouring trick

- Colours are sequences of Δ offers, which are rational numbers
- Assume that node weights are integers 1, 2, ..., W
- Then there are at most $W(\Delta!)^{\Delta}$ possible offers
- And hence only $k = (W(\Delta!)^{\Delta})^{\Delta}$ possible colours



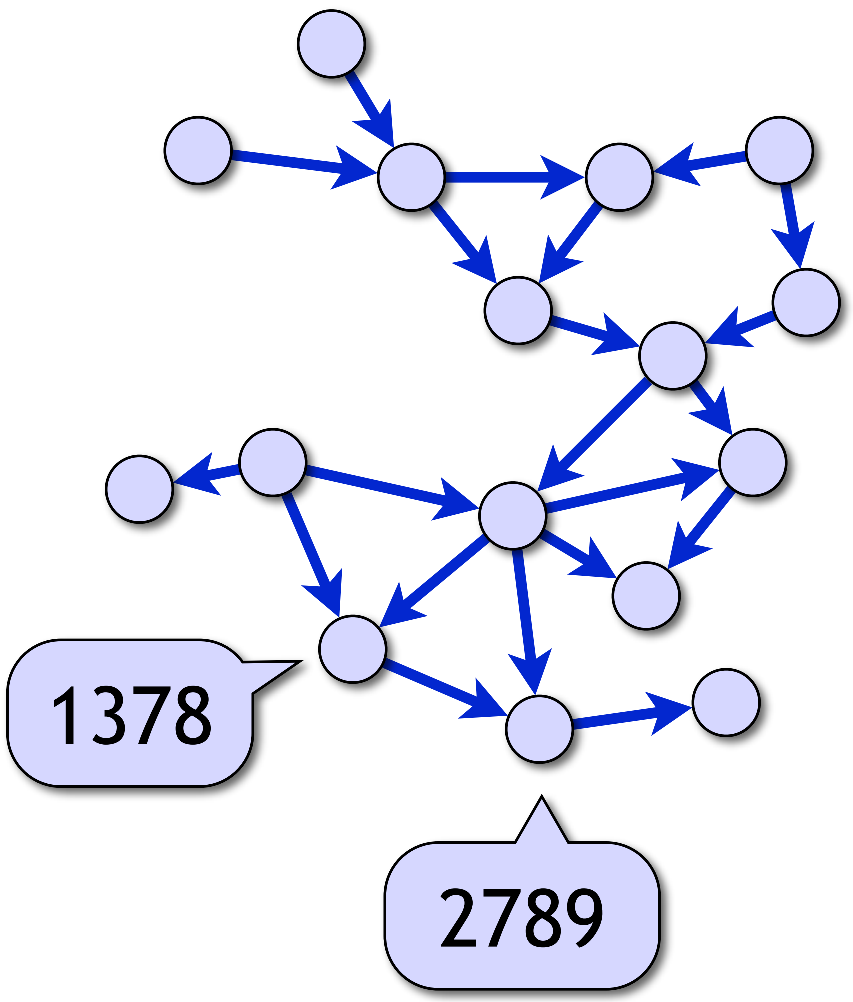
Finding a maximal edge packing: colouring trick

- Only $k = (W(\Delta!)^{\Delta})^{\Delta}$ possible colours
- Replace “inconvenient” colours (sequences of rationals) with “convenient” colours (integers 1, 2, ..., k)



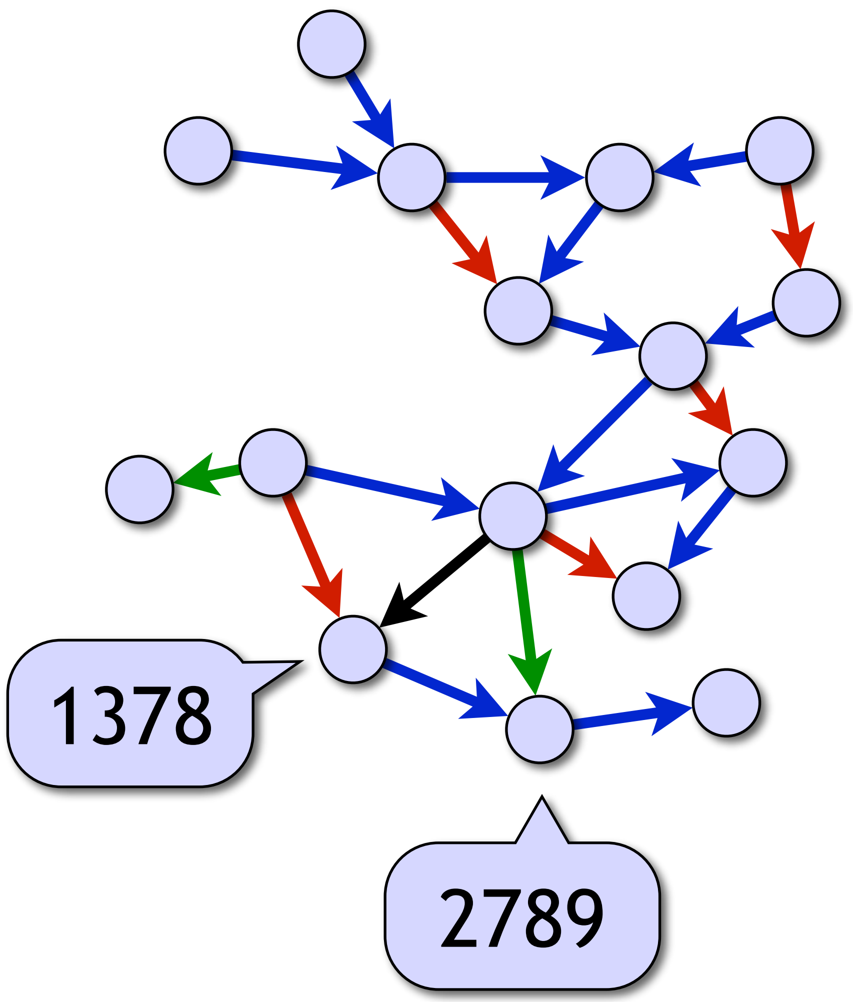
Finding a maximal edge packing: phase II

- We have a proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour (acyclic directed graph)



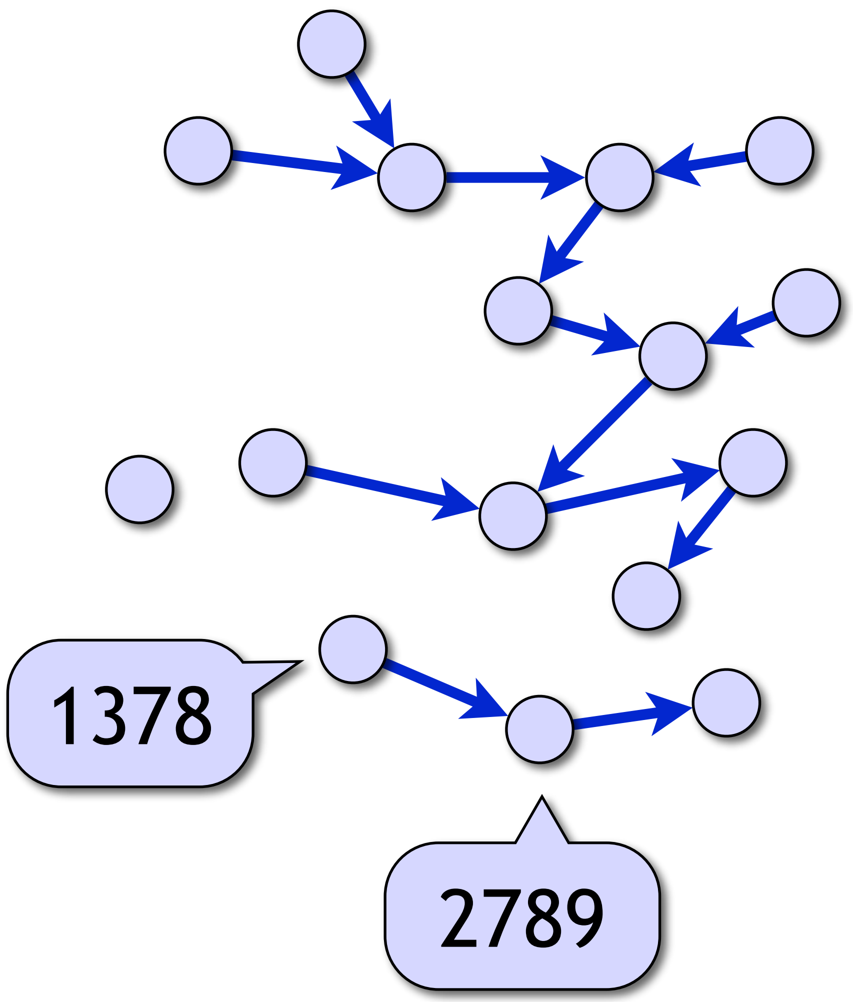
Finding a maximal edge packing: phase II

- We have a proper *k*-colouring of the unsaturated subgraph
- Orient from lower to higher colour (acyclic directed graph)
- Partition in Δ forests
  - Each node assigns its outgoing edges to different forests



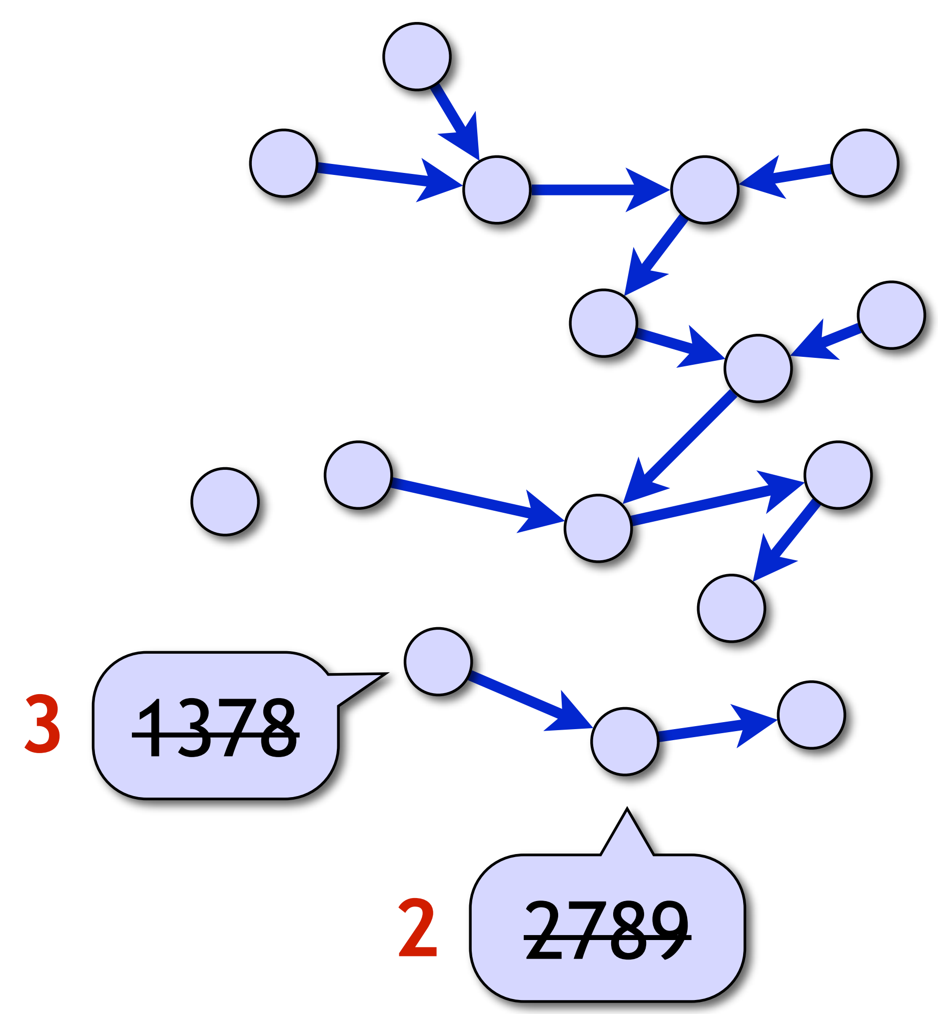
Finding a maximal edge packing: phase II

- For each forest in parallel...



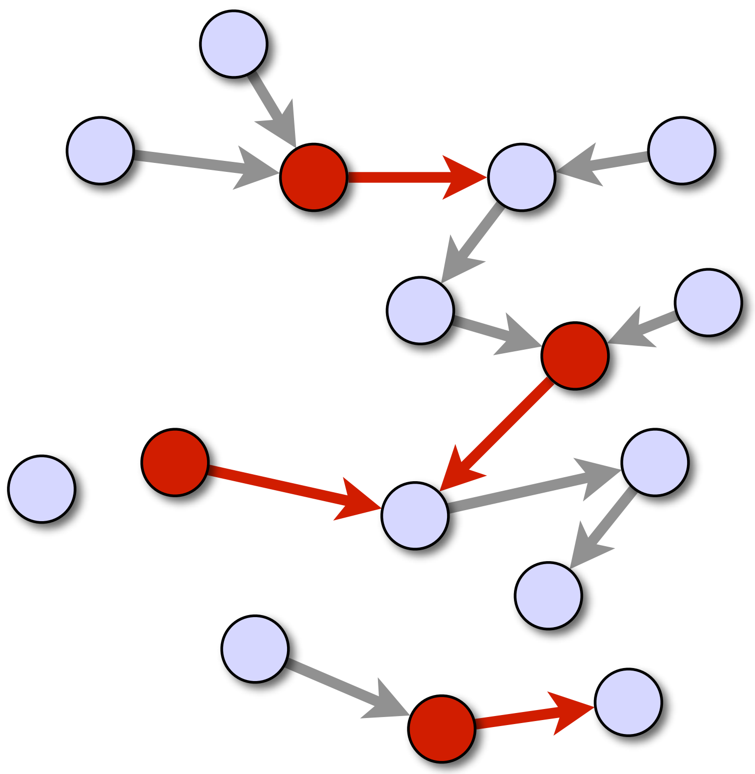
Finding a maximal edge packing: phase II

- For each forest in parallel:
  - Use Cole–Vishkin style colour reduction algorithm
  - Given a *k*-colouring, finds a **3-colouring** in time O(log* *k*)



Finding a maximal edge packing: phase II

- For each forest and each colour j = 1, 2, 3 in sequence:
  - Consider all outgoing edges of colour-j nodes



Finding a maximal edge packing: phase II

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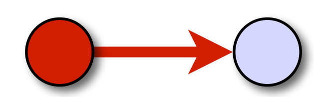
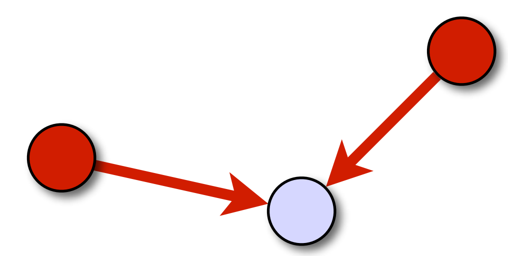
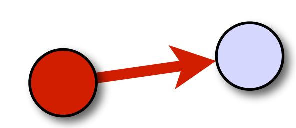


  - Node-disjoint stars: easy to saturate all such edges in parallel

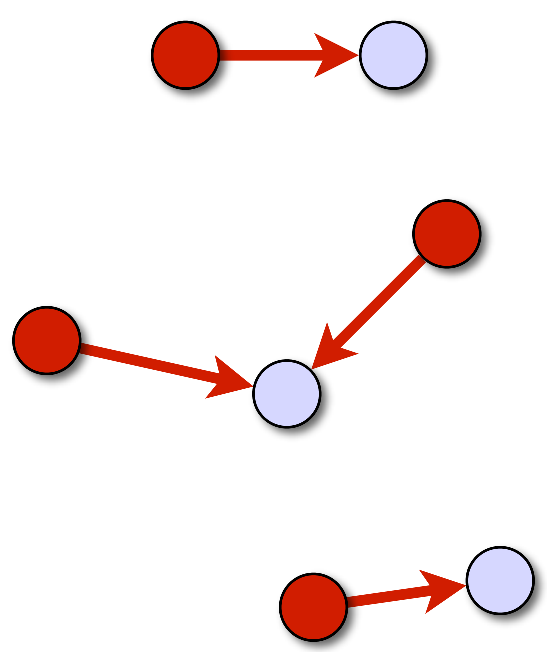


  - Two simple cases:
    - saturate centre
    - saturate all leaves



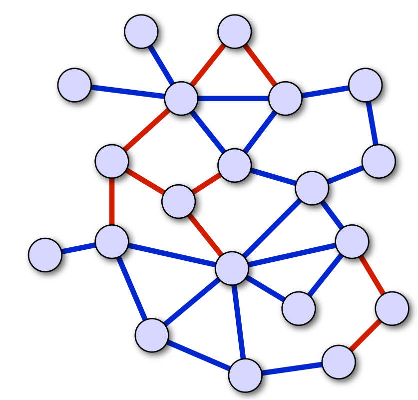
Finding a maximal edge packing: phase II

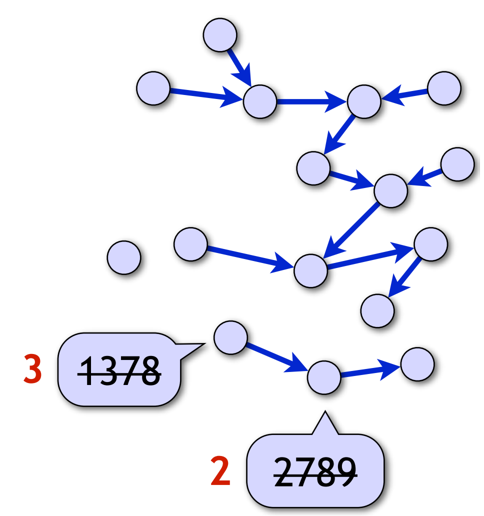
- This way we can saturate all multicoloured edges:
  - Each edge belongs to one forest, and its tail has colour 1, 2, or 3
  - We simply go through all forests and all colours and therefore saturate everything

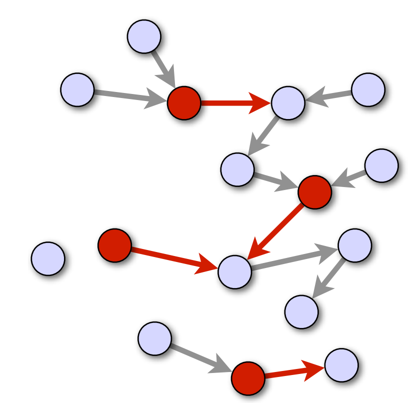


Finding a maximal edge packing: algorithm overview

- Phase I:
  - All edges become saturated or multicoloured
- Phase II:
  - Multicoloured edges are partitioned in Δ forests
  - Forests are 3-coloured
  - 3-coloured forests are saturated



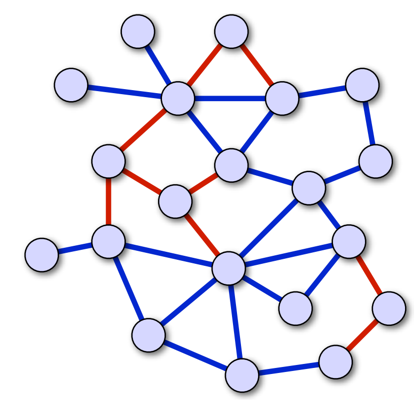


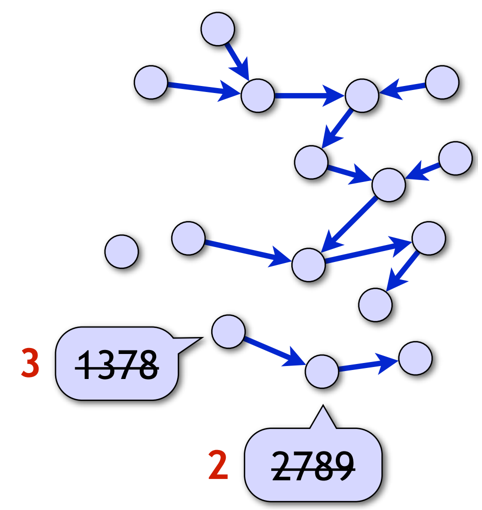


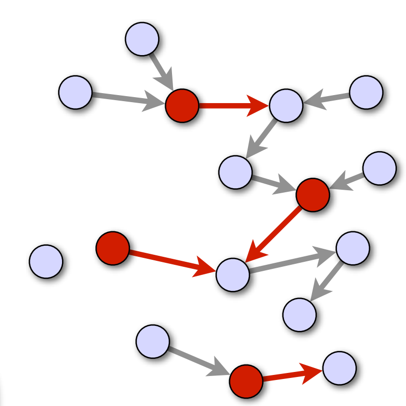
Finding a maximal edge packing: running time analysis

- Total running time:
  - All edges become saturated or multicoloured: $O(\Delta)$
  - Multicoloured forests are 3-coloured: $O(\log^* k)$
  - 3-coloured forests are saturated: $O(\Delta)$
- $O(\Delta + \log^* k) = O(\Delta + \log^* W)$

  Exercise
  - k is huge, but log* grows slowly

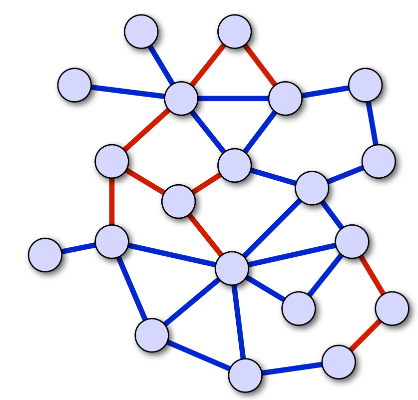


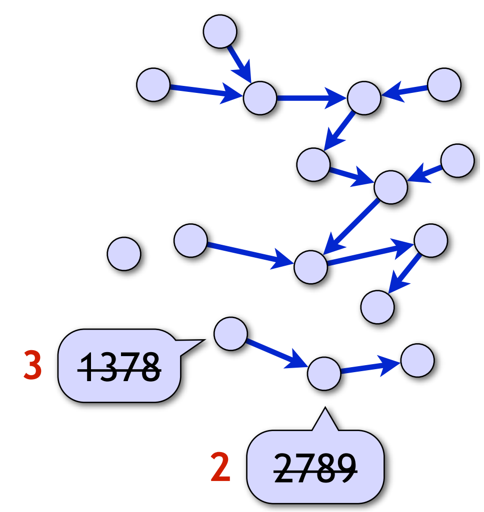


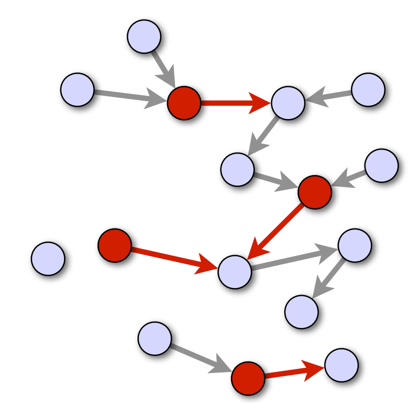


Finding a maximal edge packing: summary

- Maximal edge packing and 2-approximation of vertex cover in time $O(\Delta + \log^* W)$
  - W = maximum node weight
- Unweighted graphs: running time simply $O(\Delta)$, independent of n
- Everything can be implemented in the **port-numbering model**



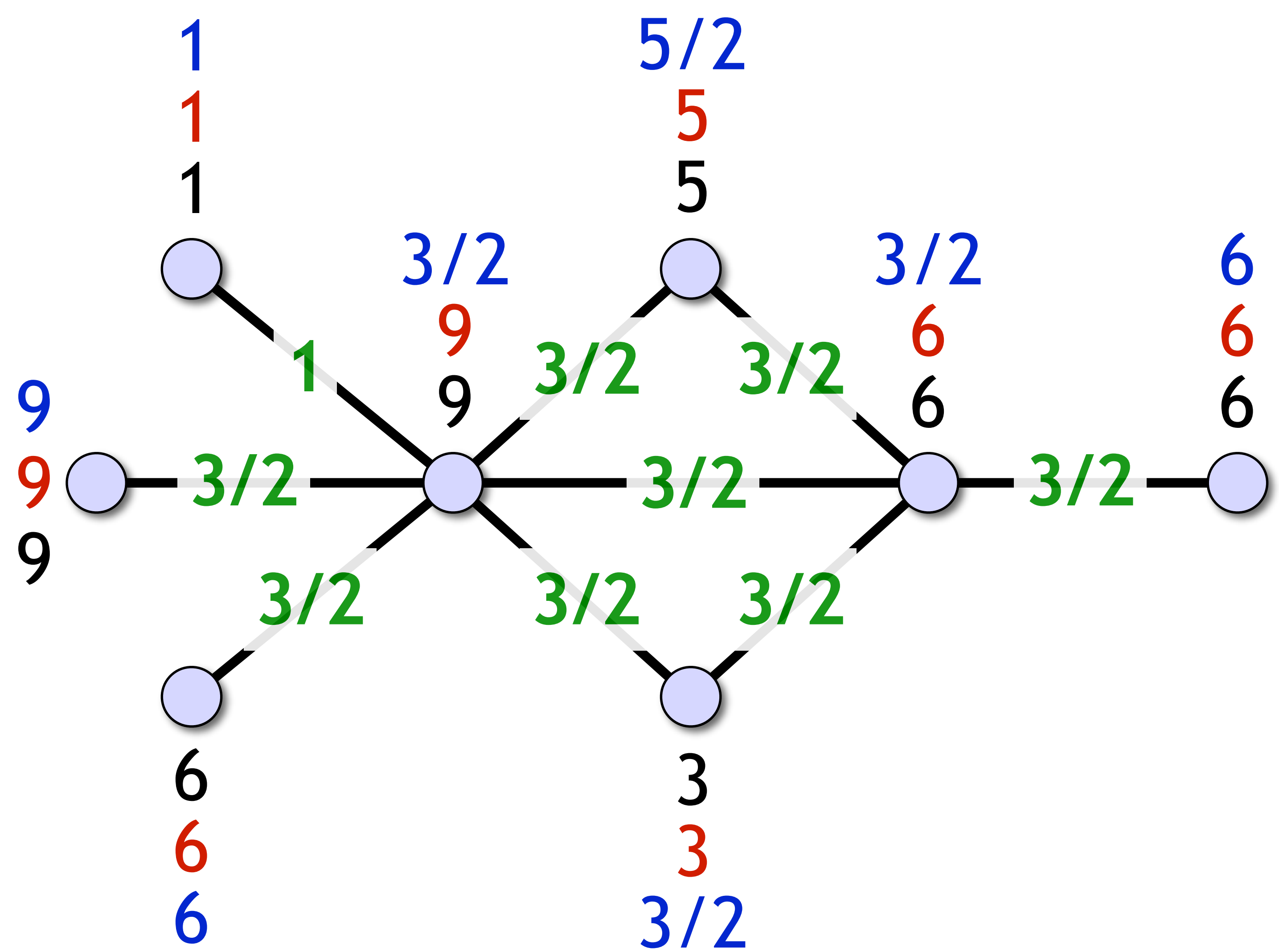




Finding a maximal edge packing: recap

Phase I:

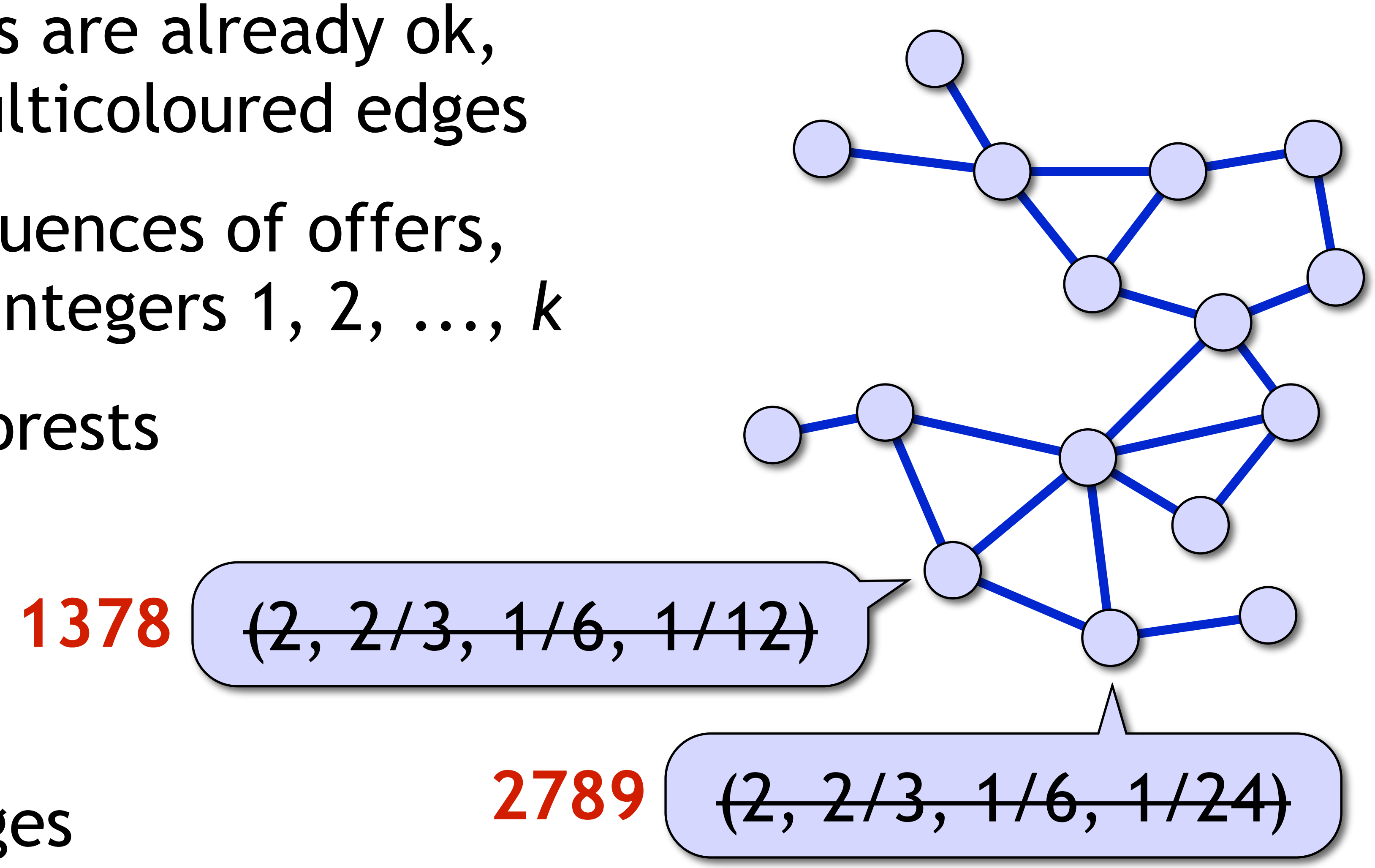
- **Residuals**
  $r(v) = w(v) - y[v]$
- **Offer** $r(v)/\deg(v)$
- **Accept minimum**, increase weights
- Progress: edges become ***saturated*** or ***multicoloured*** (different offers)



Finding a maximal edge packing: recap

Phase II:

- Saturated edges are already ok, we focus on multicoloured edges
- Colours are sequences of offers, re-colour with integers 1, 2, ..., *k*
- Partition in Δ forests
- Cole–Vishkin: 3-colouring
- Use colours to saturate all edges



Finding a maximal edge packing: some intuition

- Regular graph with uniform weights:
  - Symmetry-breaking (e.g., graph colouring) is not possible in the port-numbering model
  - But it is trivial to find a maximal edge packing directly
- "Irregular" graph:
  - We have symmetry-breaking information, which can be used to find a graph colouring, which can be used to find a maximal edge packing
- Handling these two cases turns out to be enough!

Take-home messages

- Non-trivial problems can be solved in very restrictive models of distributed computing
- Generalise!
  - More difficult problems may be easier to solve: vertex cover → weighted vertex cover → weighted set cover...
- Cole–Vishkin technique is a powerful tool
  - Wide range of applications far beyond the textbook examples of colouring cycles with numerical IDs
  - log* of almost everything is something reasonable