Chapter 6

Shared Objects

Assume that there is a common resource (e.g., a common variable or data structure), which different nodes in a network need to access from time to time. If the nodes are allowed to change the common object when accessing it, we need to guarantee that no two nodes have access to the object at the same time. In order to achieve this mutual exclusion, we need protocols that allow the nodes of a network to store and manage access to such a shared object.

6.1 Centralized Solutions

A simple and obvious solution is to store the shared object at a central location (see Algorithm 27).

Algorithm 27 Shared Object: Centralized Solution

Initialization: Shared object stored at root node $r$ of a spanning tree of the network graph (i.e., each node knows its parent in the spanning tree).

Accessing Object: (by node $v$)
1. $v$ sends request up the tree
2. Request processed by root $r$ (atomically)
3. Result sent down the tree to node $v$

Remarks:
- Instead of a spanning tree, one can use routing.
- Algorithm 27 works, but it is not very efficient. Assume that the object is accessed by a single node $v$ repeatedly. Then we get a high message/time complexity. Instead $v$ could store the object, or at least cache it. But then, in case another node stores the object, we might run into consistency problems.
- Alternative idea: The accessing node should become the new master of the object. The shared object then becomes mobile. There exist several variants of this idea. The simplest version is a home-based solution like in Mobile IP (see Algorithm 28).

6.2 Arrow and Friends

We will now look at a protocol (called the Arrow algorithm) that always moves the shared object to the node currently accessing it without creating the triangular routing problem of home-based solutions. The protocol runs on a precomputed spanning tree. Assume that the spanning tree is rooted at the current position of the shared object. When a node $u$ wants to access the shared object, it sends out a find request towards the current position of the object. While searching for the object, the edges of the spanning tree are redirected such that in the end, the spanning tree is rooted at $u$ (i.e., the new holder of the object).

Algorithm 28 Shared Object: Home-Based Solution

Initialization: An object has a home base (a node) that is known to every node. All requests (accesses to the shared object) are routed through the home base.

Accessing Object: (by node $v$)
1. $v$ acquires a lock at the home base, receives object.

Remarks:
- Home-based solutions suffer from the triangular routing problem. If two close-by nodes take turns to access the object, all the traffic is routed through the potentially far away home-base.

Theorem 6.1 (Arrow, Analysis) In an asynchronous and concurrent setting, a “find” operation terminates with message and time complexity $D$, where $D$ is the diameter of the spanning tree.
Algorithm 29 Shared Object: Arrow Algorithm

Initialization: As for Algorithm 27, we are given a rooted spanning tree. Each node has a pointer to its parent, the root of its own parent. The variable is initially stored at \( r \). For all nodes \( w \), \( w\.\text{successor} := \text{null} \), \( w\.\text{wait} := \text{false} \).

Start Find Request at Node \( u \):
1. do atomically
2. \( u \) sends "find by \( u \)" message to parent node
3. \( u\.\text{parent} := u \)
4. \( u\.\text{wait} := \text{true} \)
5. end do

Upon \( w \) Receiving "Find by \( u \)" Message from Node \( v \):
6. do atomically
7. if \( w\.\text{parent} \neq w \) then
8. \( w \) sends "find by \( u \)" message to parent
9. \( w\.\text{parent} := v \)
10. else
11. \( w\.\text{parent} := u \)
12. if not \( w\.\text{wait} \) then
13. send variable to \( u \)  // \( w \) holds var. but does not need it any more
14. else
15. \( w\.\text{successor} := u \)  // \( w \) will send variable to \( u \) a.s.a.p.
16. end if
17. end if
18. end do

Upon \( w \) Receiving Shared Object:
19. perform operation on shared object
20. do atomically
21. \( w\.\text{wait} := \text{false} \)
22. if \( w\.\text{successor} \neq \text{null} \) then
23. send variable to \( w\.\text{successor} \)
24. \( w\.\text{successor} := \text{null} \)
25. end if
26. end do

Before proving Theorem 6.1, we prove the following lemma.

Lemma 6.2. An edge \( \{u, v\} \) of the spanning tree \( T \) is in one of four states:

1.) Pointer from \( u \) to \( v \) (no message on the edge, no pointer from \( v \) to \( u \))
2.) Message on the more from \( u \) to \( v \) (no pointer along the edge)
3.) Pointer from \( v \) to \( u \) (no message on the edge, no pointer from \( u \) to \( v \))
4.) Message on the more from \( v \) to \( u \) (no pointer along the edge)

Proof. W.l.o.g., assume that initially the edge \( \{u, v\} \) is in state 1. With a message arrival at \( u \) (or \( u \) starts a "find by \( u \)" request, the edge goes to state 2. When the message is received at \( v \), \( v \) directs its pointer to \( u \) and we are therefore in state 3. A new message at \( v \) (or a new request initiated by \( v \)) then brings the edge back to state 1.

Proof of Theorem 6.1. Since the "find" message will only travel on a static tree, it suffices to show that it will not traverse an edge twice. Suppose for the sake of contradiction that there is a first "find" message \( f \) that traverses an edge \( e = \{u, v\} \) for the second time and assume that \( e \) is the first edge that is traversed twice by \( f \). The first time, \( f \) traverses \( e \). Assume that \( e \) is first traversed from \( u \) to \( v \). Since we are on a tree, the second time, \( e \) must be traversed from \( v \) to \( u \). Because \( e \) is the first edge to be traversed twice, \( f \) must re-visit \( e \) before visiting any other edges. Right before \( f \) reaches \( v \), the edge \( e \) is in state 2 (\( f \) is on the move) and in state 3 (it will immediately return with the pointer from \( v \) to \( u \)). This is a contradiction to Lemma 6.2.

Remarks:
- Finding a good tree is an interesting problem. We would like to have a tree with low stretch, low diameter, low degree, etc.
- It seems that the Arrow algorithm works especially well when lots of "find" operations are initiated concurrently. Most of them will find a "close-by" node, thus having low message/time complexity. For the sake of simplicity we analyze a synchronous system.

Theorem 6.3. (Arrow, Concurrent Analysis) Let the system be synchronous. Initially, the system is in a quiescent state. At time 0, a set \( S \) of nodes initiates a "find" operation. The message complexity of all "find" operations is \( O(\log |S|) \) where \( m^* \) is the message complexity of an optimal (with global knowledge) algorithm on the tree.

Proof Sketch. Let \( d \) be the minimum distance of any node in \( S \) to the root. There will be a node \( u_1 \) at distance \( d \) from the root that reaches the root in \( d \) time steps, turning all the arrows on the path to the root towards \( u_1 \). A node \( u_2 \) that finds (is spawned behind) \( u_1 \) cannot distinguish the system from a system where there was no request \( u_1 \), and instead the root was initially located at \( u_1 \). The message cost of \( u_2 \) is consequentially the distance between \( u_1 \) and \( u_2 \) on the spanning tree. By induction the total message complexity is exactly as if a collector starts at the root and then "greedily" collects tokens located at the nodes in \( S \) (greedily in the sense that the collector always goes towards the closest token). Greedy collecting the tokens is not a good strategy in general because it will traverse the same edge more than twice in the worst
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case. An asymptotically optimal algorithm can also be translated into a depth-
first-search collecting paradigm, traversing each edge at most twice. In another
area of computer science, we would call the Arrow algorithm a nearest-neighbor
TSP heuristic (without returning to the start/root though), and the optimal
algorithm TSP-optimal. It was shown that nearest-neighbor has a logarithmic
overhead, which concludes the proof.

Remarks:
• An average request set \( S \) on a not-too-bad tree gives usually a much
better bound. However, there is an almost tight \( \log |S|/\log \log |S| \)
worst-case example.
• It was recently shown that Arrow can do as good in a dynamic setting
(where nodes are allowed to initiate requests at any time). In partic-
tular the message complexity of the dynamic analysis can be shown to
have a \( \log D \) overhead only, where \( D \) is the diameter of the spanning
tree (note that for logarithmic trees, the overhead becomes \( \log \log n \)).
• What if the spanning tree is a star? Then with Theorem 6.1, each find
will terminate in 2 steps! Since also an optimal algorithm has message
cost 1, the algorithm is 2-competitive. . . ? Yes, but because of its high
degree the star center experiences contention... It can be shown that
the contention overhead is at most proportional to the largest degree
\( \Delta \) of the spanning tree.
• Thought experiment: Assume a balanced binary spanning tree—by
Theorem 6.1, the message complexity per operation is \( \log n \). Because
a binary tree has maximum degree 3, the time per operation therefore
is at most \( 3 \log n \).
• There are better and worse choices for the spanning tree. The stretch
of an edge \( \{u, v\} \) is defined as distance between \( u \) and \( v \) in a span-
ing tree. The maximum stretch of a spanning tree is the maximum
stretch over all edges. A few years ago, it was shown how to construct
spanning trees that are \( O(\log n) \)-stretch-competitive.

What if most nodes just want to read the shared object? Then it does
not make sense to acquire a lock every time. Instead we can use caching (see
Algorithm 30).

Theorem 6.4. Algorithm 30 is correct. More surprisingly, the message com-
plexity is \( 3 \)-competitive (at most a factor 3 worse than the optimum).

Proof. Since the accesses do not overlap by definition, it suffices to show that
between two writes, we are \( 3 \)-competitive. The sequence of accessing nodes is
\( w_0, r_1, r_2, \ldots, r_k, u_1 \). After \( w_0 \), the object is stored at \( w_0 \) and not cached
anywhere else. All reads cost twice the smallest subtree \( T \) spanning the write
\( w_0 \) and all the reads since each read only goes to the first copy. The write \( u_1 \)
costs \( T \) plus the path \( P \) from \( w_0 \) to \( T \). Since any data management scheme
must use an edge in \( T \) and \( P \) at least once, and our algorithm uses edges in \( T \)
at most 3 times (and in \( P \) at most once), the theorem follows.

Algorithm 30 Shared Object: Read/Write Caching

- Nodes can either read or write the shared object. For simplicity we first
  assume that reads or writes do not overlap in time (access to the object is
  sequential).
- Nodes store three items: a parent pointer pointing to one of the neighbors
  (as with Arrow), and a cache bit for each edge, plus (potentially) a copy of
  the object.
- Initially the object is stored at a single node \( u \); all the parent pointers point
  towards \( u \), all the cache bits are false.
- When initiating a read, a message follows the arrows (this time: without
  inverting them!) until it reaches a cached version of the object. Then a copy
  of the object is cached along the path back to the initiating node, and the
  cache bits on the visited edges are set to true.
- A write at \( u \) writes the new value locally (at node \( u \)), then searches (follow
  the parent pointers and reverse them towards \( u \)) a first node with a copy. Delete
  the copy and follow (in parallel, by flooding) all edge that have the cache flag
  set. Point the parent pointer towards \( u \), and remove the cache flags.

Remarks:
• Concurrent reads are not a problem, also multiple concurrent reads
  and one write work just fine.
• What about concurrent writes? To achieve consistency writes need to
  invalidate the caches before writing their value. It is claimed that the
  strategy then becomes \( 4 \)-competitive.
• Is the algorithm also time competitive? Well, not really: The optimal
  algorithm that we compare to is usually offline. This means it knows
  the whole access sequence in advance. It can then cache the object
  before the request even appears!
• Algorithms on trees are often simpler, but have the disadvantage that
  they introduce the extra stretch factor. In a ring, for example, any
tree has stretch \( n - 1 \); so there is always a bad request pattern.
Algorithm 31 Shared Object: Pointer Forwarding

Initialization: Object is stored at root \( r \) of a precomputed spanning tree \( T \) (as in the Arrow algorithm, each node has a parent pointer pointing towards the object).

Accessing Object: (by node \( u \))
1. follow parent pointers to current root \( r \) of \( T \)
2. send object from \( r \) to \( u \)
3. \( u \), parent := \( u \); \( u \), parent := \( u \); // \( u \) is the new root

Algorithm 32 Shared Object: Ivy

Initialization: Object is stored at root \( r \) of a precomputed spanning tree \( T \) (as before, each node has a parent pointer pointing towards the object). For simplicity, we assume that accesses to the object are sequential.

Start Find Request at Node \( u \):
1. \( u \) sends “find by \( u \)” message to parent node
2. \( u \), parent := \( u \)

Upon \( v \) receiving “Find by \( u \)” Message:
3. if \( v \), parent = \( v \) then
4. send object to \( u \)
5. else
6. send “find by \( u \)” message to \( v \), parent
7. end if
8. \( v \), parent := \( u \); // \( u \) will become the new root

6.3 Ivy and Friends

In the following we study algorithms that do not restrict communication to a tree. Of particular interest is the special case of a complete graph (clique). A simple solution for this case is given by Algorithm 31.

Remarks:
- If the graph is not complete, routing can be used to find the root.
- Assume that the nodes line up in a linked list. If we always choose the first node of the linked list to acquire the object, we have message/time complexity \( n \). The new topology is again a linked list. Pointer forwarding is therefore bad in a worst-case.
- If edges are not FIFO, it can even happen that the number of steps is unbounded for a node having bad luck. An algorithm with such a property is named “not fair,” or “not wait-free.” (Example: Initially we have the list \( 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \); 4 starts a find, when the message of 4 passes 3, 3 itself starts a find. The message of 3 may arrive at 2 and then 1 earlier, thus the new end of the list is \( 2 \rightarrow 1 \rightarrow 3 \); once the message of 4 passes 2, the game re-starts.)

There seems to be a natural improvement of the pointer forwarding idea. Instead of simply redirecting the parent pointer from the old root to the new root, we can redirect all the parent pointers of the nodes on the path visited during a find message to the new root. The details are given by Algorithm 32.

Figure 6.1 shows how the pointer redirecting affects a given tree (the right tree results from a find request started at node \( x_0 \) on the left tree).

Remarks:
- Also with Algorithm 32, we might have a bad linked list situation. However, if the start of the list acquires the object, the linked list turns into a star. As the following theorem shows, the search paths are not long on average. Since paths sometimes can be bad, we will need amortized analysis.

Theorem 6.5. If the initial tree is a star, a find request of Algorithm 32 needs at most \( \log n \) steps on average, where \( n \) is the number of processors.

Proof. All logarithms in the following proof are to base 2. We assume that accesses to the shared object are sequential. We use a potential function argument. Let \( s(u) \) be the size of the subtree rooted at node \( u \) (the number of nodes in the subtree including \( u \) itself). We define the potential \( \Phi \) of the whole tree \( T \) as \( (V \) is the set of all nodes)

\[
\Phi(T) = \sum_{u \in V} \log s(u) / 2
\]

Assume that the path traversed by the \( i \)th operation has length \( k_i \), i.e., the \( i \)th operation redirects \( k_i \) pointers to the new root. Clearly, the number of steps of the \( i \)th operation is proportional to \( k_i \). We are interested in the cost of \( m \) consecutive operations, \( \sum_{i=1}^{m} k_i \).

Let \( T_0 \) be the initial tree and let \( T_i \) be the tree after the \( i \)th operation. Further, let \( a_i = k_i - \Phi(T_{i-1}) + \Phi(T_i) \) be the amortized cost of the \( i \)th operation.

We have

\[
\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} (k_i - \Phi(T_{i-1}) + \Phi(T_i)) = \sum_{i=1}^{m} k_i - \Phi(T_0) + \Phi(T_m).
\]

For any tree \( T \), we have \( \Phi(T) \geq \log(n)/2 \). Because we assume that \( T_0 \) is a star, we also have \( \Phi(T_0) = \log(n)/2 \). We therefore get that

\[
\sum_{i=1}^{m} a_i \geq \sum_{i=1}^{m} k_i.
\]
Hence, it suffices to upper bound the amortized cost of every operation. We thus analyze the amortized cost $a_i$ of the $i^{th}$ operation. Let $x_0, x_1, x_2, \ldots, x_k$ be the path that is reversed by the operation. Further for $0 \leq j \leq k$, let $s_j$ be the size of the subtree rooted at $x_j$ before the reversal. The size of the subtree rooted at $x_0$ after the reversal is $s_k$, and the size of the one rooted at $x_j$ after the reversal, for $1 \leq j \leq k$, is $s_{j-1}$ (see Figure 6.1). For all other nodes, the sizes of their subtrees are the same, therefore the corresponding terms cancel out in the amortized cost $a_i$. We can thus write $a_i$ as

$$a_i = k_i + \frac{1}{2} \sum_{j=0}^{k_i-1} \log(s_{j} - s_{j-1}) + \frac{1}{2} \log s_{k_i} + \frac{1}{2} \sum_{j=1}^{k_i} \log(s_j - s_{j-1})$$

$$= k_i + \frac{1}{2} \sum_{j=0}^{k_i-1} \log(s_{j+1} - s_j) - \log s_j$$

$$= k_i + \frac{1}{2} \sum_{j=0}^{k_i-1} \log \frac{s_{j+1} - s_j}{s_j}.$$ 

For $0 \leq j \leq k_i - 1$, let $a_j = s_{j+1}/s_j$. Note that $s_{j+1} > s_j$ and thus that $a_j > 1$. Further note, that $(s_{j+1} - s_j)/s_j = a_j - 1$. We therefore have that

$$a_i = k_i + \frac{1}{2} \sum_{j=0}^{k_i-1} \log(a_j - 1)$$

$$= \sum_{j=0}^{k_i-1} \left(1 + \frac{1}{2} \log(a_j - 1)\right).$$

For $\alpha > 1$, it can be shown that $1 + \log(\alpha - 1)/2 \leq \log \alpha$ (see Lemma 6.6). From this inequality, we obtain

$$a_i \leq \sum_{j=0}^{k_i-1} \log a_j = \sum_{j=0}^{k_i-1} \log \frac{s_{j+1}}{s_j} = \sum_{j=0}^{k_i-1} \log \frac{s_{j+1} - s_j}{s_j}$$

$$= \log s_{k_i} - \log s_0 \leq \log n,$$

because $s_k = n$ and $k_0 \geq 1$. This concludes the proof.

Lemma 6.6. For $\alpha > 1$, $1 + \log(\alpha - 1)/2 \leq \log \alpha$.

Proof. The claim can be verified by the following chain of reasoning:

$$0 \leq (\alpha - 2)^2$$

$$0 \leq \alpha^2 - 4\alpha + 4$$

$$4(\alpha - 1) \leq \alpha^2$$

$$\log_2 4(\alpha - 1) \leq \log_2 (\alpha^2)$$

$$2 + \log_2(\alpha - 1) \leq 2 \log_2 \alpha$$

$$1 + \frac{1}{2} \log_2(\alpha - 1) \leq \log_2 \alpha.$$
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