DDA 2010, lecture 3: Ramsey's theorem

- A generalisation of the pigeonhole principle
- Frank P. Ramsey (1930):
  "On a problem of formal logic"
  - "… in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest…"

DDA 2010, lecture 3a: Introduction to Ramsey's theorem

- Notation of Ramsey numbers from Radziszowski (2009)

Basic definitions

- Assign a colour from {1, 2, …, c} to each k-subset of {1, 2, …, N}

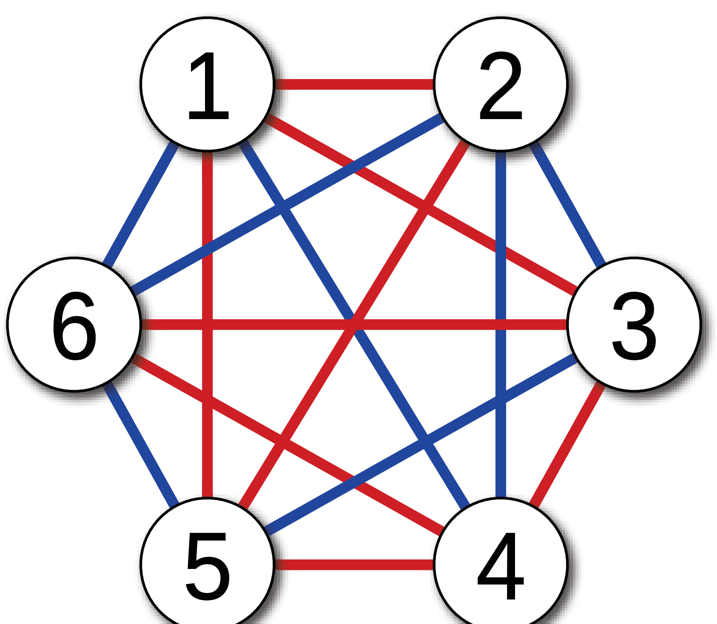
N = 4, k = 3, c = 2	
{1,2,3}	{1,2,4}
{1,3,4}	{2,3,4}

N = 13, k = 1, c = 3			
{1}	{2}	{3}	{4}
{5}	{6}	{7}	{8}
{9}	{10}	{11}	{12}
{13}			

N = 6, k = 2, c = 2				
{1,2}	{1,3}	{1,4}	{1,5}	{1,6}
	{2,3}	{2,4}	{2,5}	{2,6}
		{3,4}	{3,5}	{3,6}
			{4,5}	{4,6}
				{5,6}

Basic definitions

- Assign a colour from $\{1, 2, \ldots, c\}$ to each k-subset of $\{1, 2, \ldots, N\}$



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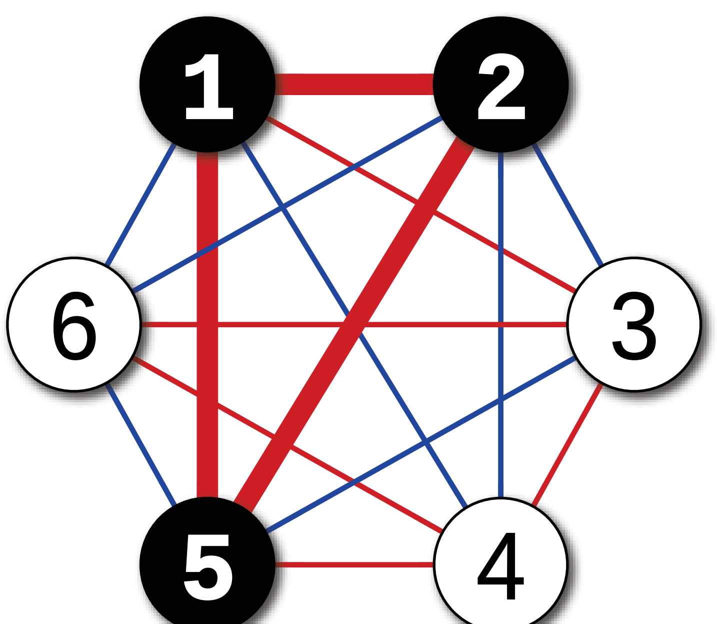
$N = 6$, $k = 2$, $c = 2$				
{1,2}	{1,3}	{1,4}	{1,5}	{1,6}
	{2,3}	{2,4}	{2,5}	{2,6}
		{3,4}	{3,5}	{3,6}
			{4,5}	{4,6}
				{5,6}

Basic definitions

- $X \subset \{1, 2, \ldots, N\}$ is a ***monochromatic subset*** if all k-subsets of X have the same colour



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$N = 6$, $k = 2$, $c = 2$				
{1,2}	{1,3}	{1,4}	**{1,5}**	{1,6}
	{2,3}	{2,4}	**{2,5}**	{2,6}
		{3,4}	{3,5}	{3,6}
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Ramsey's theorem

- Assign a colour from $\{1, 2, \ldots, c\}$ to each k-subset of $\{1, 2, \ldots, N\}$
- $X \subset \{1, 2, \ldots, N\}$ is a monochromatic subset if all k-subsets of X have the same colour
- **Ramsey's theorem**: For all c, k, and n there is a finite N such that ***any*** c-colouring of k-subsets of $\{1, 2, \ldots, N\}$ contains a monochromatic subset with n elements

Ramsey's theorem

- Assign a colour from $\{1, 2, \ldots, c\}$ to each k-subset of $\{1, 2, \ldots, N\}$
- $X \subset \{1, 2, \ldots, N\}$ is a monochromatic subset if all k-subsets of X have the same colour
- **Ramsey's theorem**: For all c, k, and n there is a finite N such that ***any*** c-colouring of k-subsets of $\{1, 2, \ldots, N\}$ contains a monochromatic subset with n elements
  - The smallest such N is denoted by $R_c(n; k)$

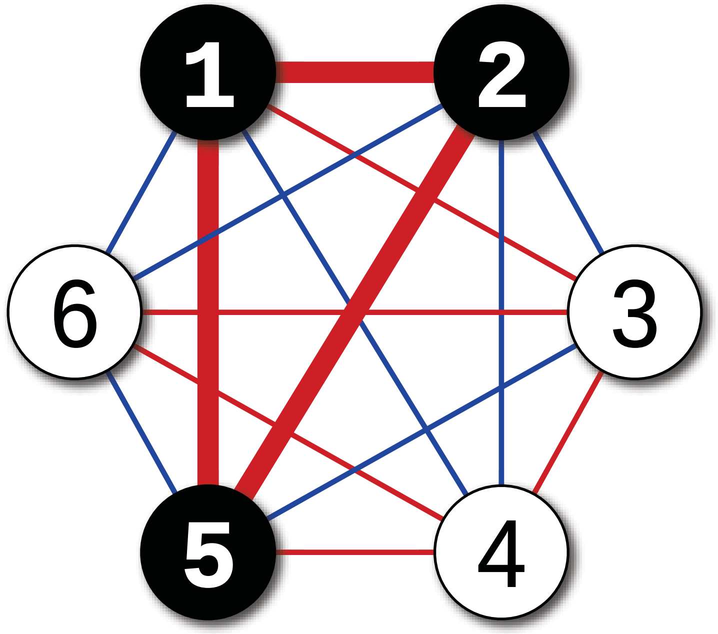
Ramsey numbers

Ramsey's theorem: $k = 1$

- $k = 1$: pigeonhole principle
- If we put N items into c slots, then at least one of the slots has to contain at least n items
  - Colour of the 1-subset $\{i\}$ = slot of the element i
  - Clearly holds if $N \geq c(n - 1) + 1$
  - Does not necessarily hold if $N \leq c(n - 1)$
  - $R_c(n; 1) = c(n - 1) + 1$

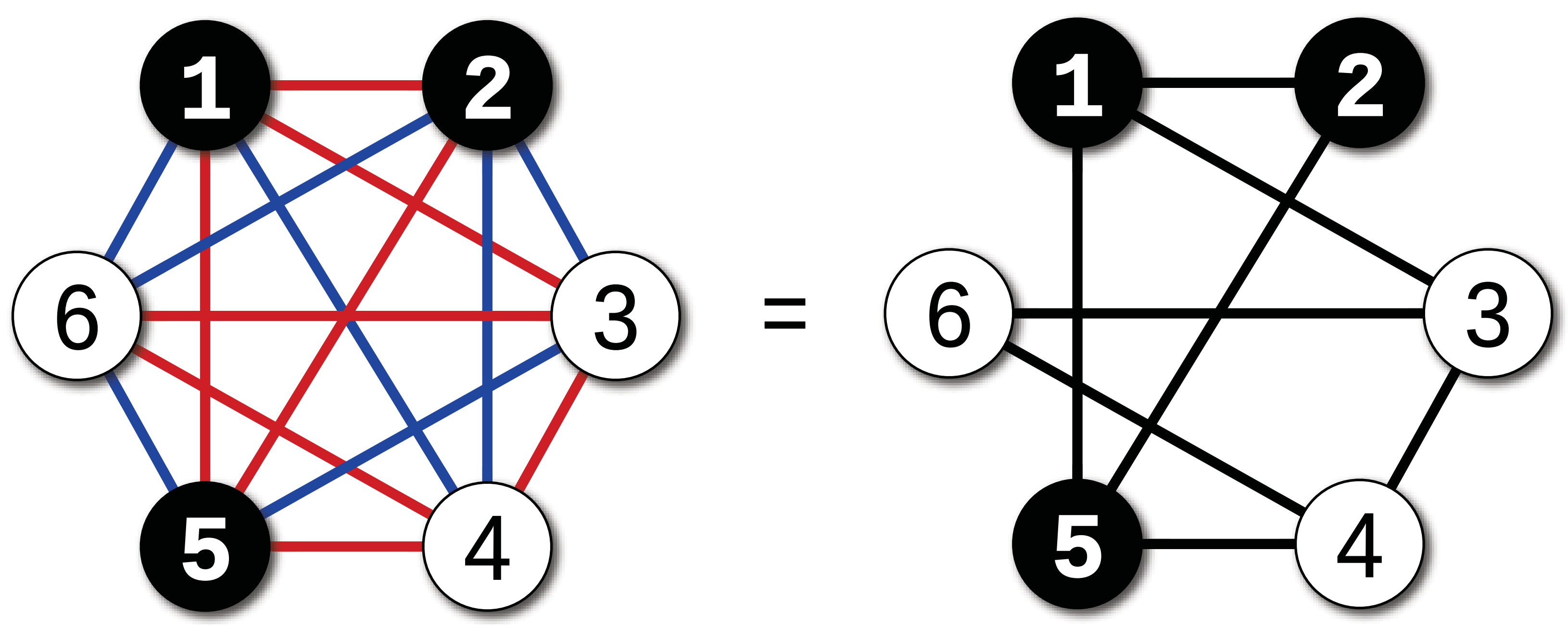
Ramsey's theorem: $k = 2$, $c = 2$

- ***Complete graphs***, red and blue edges
- If the graph is large enough, there will be a ***monochromatic clique***
  - For example, $R_2(2; 2) = 2$, $R_2(3; 2) = 6$, and $R_2(4; 2) = 18$
  - A graph with 2 nodes contains a monochromatic edge
  - A graph with 6 nodes contains a monochromatic triangle



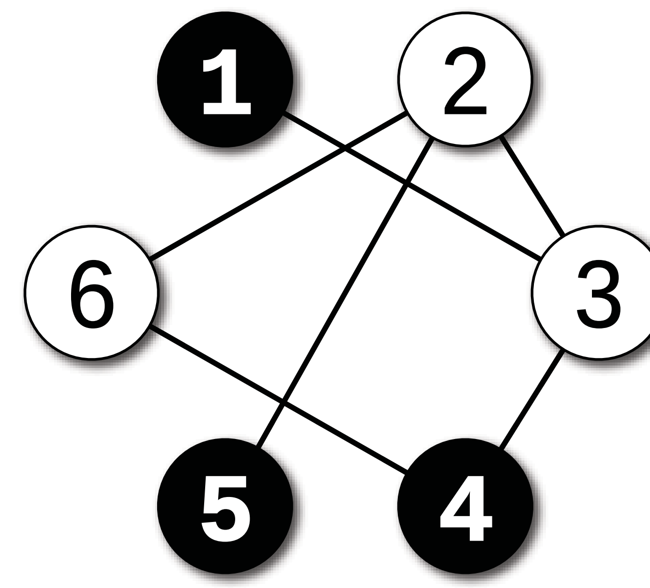
Ramsey's theorem: $k = 2$, $c = 2$

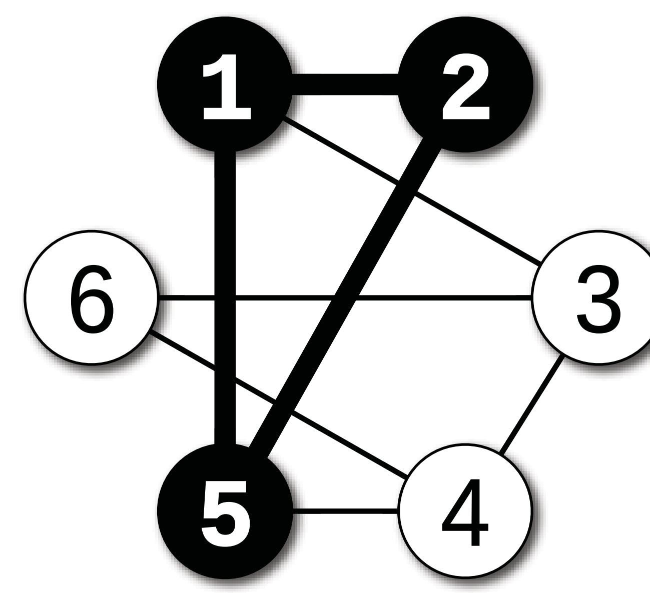
- Of course, we can equally well have:
  - red / blue edges
  - existing / missing edges



Ramsey's theorem: $k = 2$, $c = 2$

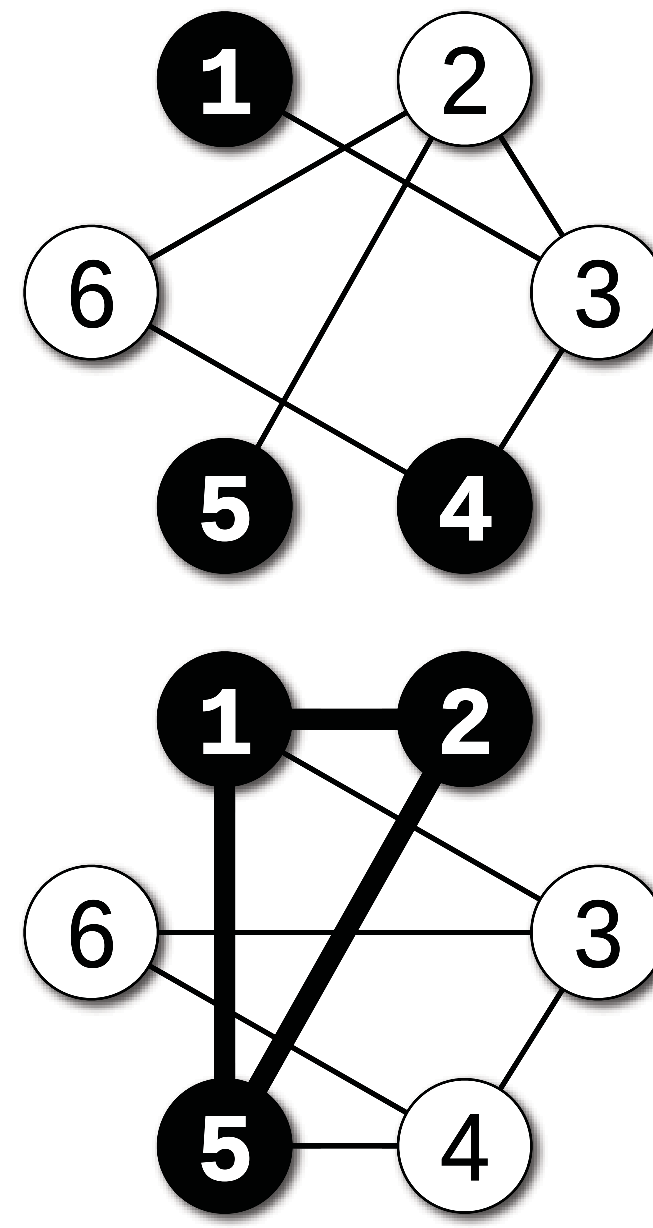
- Another interpretation: graphs
  - $\{u,v\}$ red: edge $\{u,v\}$ present
  - $\{u,v\}$ blue: edge $\{u,v\}$ missing
- Large monochromatic subset:
  - Large clique (red) or large independent set (blue)
  - Any graph with 6 nodes contains a clique with 3 nodes or an independent set with 3 nodes





Ramsey’s theorem: $k = 2$, $c = 2$

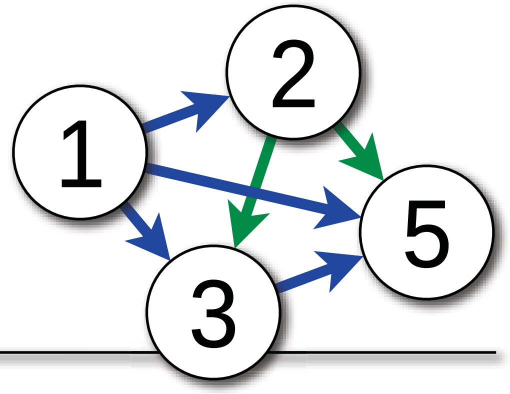
- Sufficiently large graphs (N nodes) contain large ***independents sets*** (n nodes) or large ***cliques*** (n nodes)
  - You can avoid one of these, but not both
  - However, Ramsey numbers are large: here N is exponential in n



DDA 2010, lecture 3b: Proof of Ramsey’s theorem

- Following Nešetřil (1995)
- Notation from Radziszowski (2009)

Definitions



- X ⊂ {1, 2, …, *N*} is a ***monochromatic subset***:
  if *A* and *B* are *k*-subsets of *X*,
  then *A* and *B* have the same colour
- X ⊂ {1, 2, …, *N*} is a ***good subset***:
  if *A* and *B* are *k*-subsets of *X* and min(*A*) = min(*B*),
  then *A* and *B* have the same colour
  - An example with *c* = 2 and *k* = 2:
    {1,2,3,5} is good but not monochromatic in the colouring
    {1,2}, **{1,3}**, {1,4}, **{1,5}**, **{2,3}**, {2,4}, **{2,5}**, **{3,5}**, {4,5}

Definitions

- $X \subset \{1, 2, \ldots, N\}$ is a ***monochromatic subset***:
  if A and B are k-subsets of X,
  then A and B have the same colour
- $X \subset \{1, 2, \ldots, N\}$ is a ***good subset***:
  if A and B are k-subsets of X and $\min(A) = \min(B)$,
  then A and B have the same colour
  - $R_c(n; k)$ = smallest N s.t. ∃ monochromatic n-subset
  - $G_c(n; k)$ = smallest N s.t. ∃ good n-subset

Proof outline

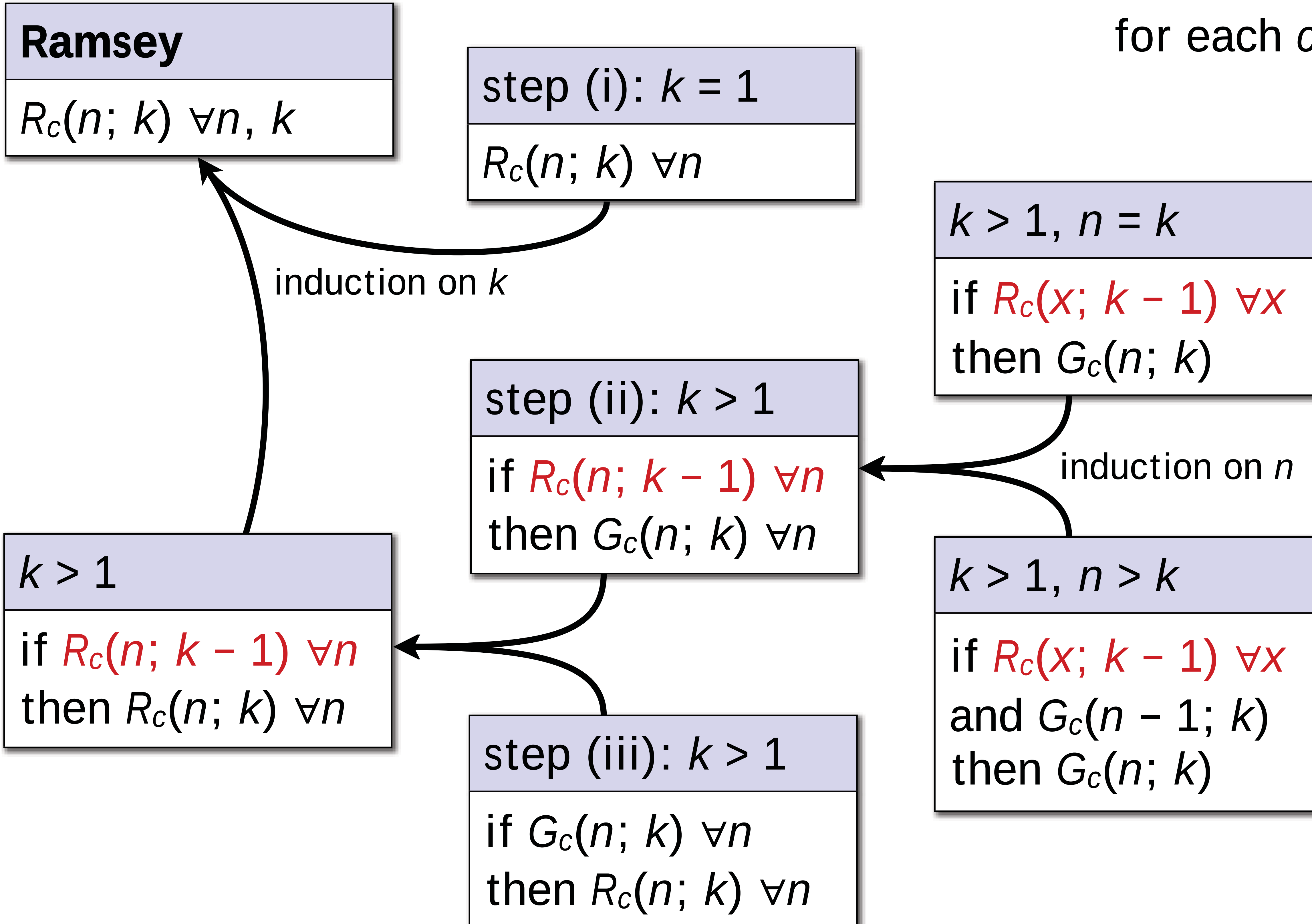
- $R_c(n; k)$ = smallest N s.t. ∃ monochromatic n-subset
- $G_c(n; k)$ = smallest N s.t. ∃ good n-subset
- **Theorem**: $R_c(n; k)$ is finite for all c, n, k

  (i) $R_c(n; 1)$ is finite for all n

  (ii) If $R_c(n; k - 1)$ is finite for all n then $G_c(n; k)$ is finite for all n

  (iii) $R_c(n; k) \leq G_c(c(n - 1) + 1; k)$ for all n, k

c is fixed throughout the proof



for each c
Ramsey
$R_c(n; k)\ \forall n, k$
step (i): $k = 1$
$R_c(n; k)\ \forall n$
induction on k
$k > 1, n = k$
if $R_c(x; k-1)\ \forall x$
then $G_c(n; k)$
step (ii): $k > 1$
if $R_c(n; k-1)\ \forall n$
then $G_c(n; k)\ \forall n$
induction on n
$k > 1$
if $R_c(n; k-1)\ \forall n$
then $R_c(n; k)\ \forall n$
$k > 1, n > k$
if $R_c(x; k-1)\ \forall x$
and $G_c(n-1; k)$
then $G_c(n; k)$
step (iii): $k > 1$
if $G_c(n; k)\ \forall n$
then $R_c(n; k)\ \forall n$

Proof: step (i)

- **Lemma**: $R_c(n; 1)$ is finite for all n
- **Proof**:
  - Pigeonhole principle
  - $R_c(n; 1) = c(n - 1) + 1$

Proof: step (ii) — outline

- **Lemma**: if $R_c(n; k - 1)$ is finite for all n then $G_c(n; k)$ is finite for all n
- **Proof**:
    - Induction on n
    - ***Basis***: $G_c(k; k)$ is finite
    - ***Inductive step***: Assume that $M = G_c(n - 1; k)$ is finite
    - Then we also have a finite $R_c(M; k - 1)$
    - Enough to show that $G_c(n; k) \leq 1 + R_c(M; k - 1)$

Proof: step (ii)

f:	{1,2,3}	{1,2,4}	{1,3,4}	{2,3,4}
f':	{2,3}	{2,4}	{3,4}	

- $G_c(n; k) \leq 1 + R_c(M; k - 1)$ where $M = G_c(n - 1; k)$
  - Let $N = 1 + R_c(M; k - 1)$, consider any colouring f of k-subsets of $\{1, 2, \ldots, N\}$
  - Delete element 1: colouring f' of $(k - 1)$-subsets of $\{2, 3, \ldots, N\}$
  - Find an f'-monochromatic M-subset $X \subset \{2, 3, \ldots, N\}$
  - Find an f-good $(n - 1)$-subset $Y \subset X$
  - $\{1\} \cup Y$ is an f-good n-subset of $\{1, 2, \ldots, N\}$

Proof: step (ii)

In real life, these constants would be much larger…

- A fictional example: $N = 7$, $M = 5$, $n = 5$, $k = 3$
  - Original colouring f: {1,2,3}, {1,2,4}, {1,2,5}, {1,2,6}, {1,2,7}, …, {1,6,7}, {2,3,4}, …, {5,6,7}
  - Colouring f': {2,3}, {2,4}, {2,5}, {2,6}, {2,7}, …, {6,7}
  - f'-monochromatic M-subset {2,3,4,5,7} of $\{2,3,\dots,N\}$: {2,3}, {2,4}, {2,5}, {2,7}, …, {5,7}
  - f-good $(n-1)$-subset {2,4,5,7}: {2,4,5}, {2,4,7}, {4,5,7}
  - {***1***,2,4,5,7} is f-good: {***1***,2,4}, {***1***,2,5}, {***1***,2,7}, …, {***1***,5,7}, {2,4,5}, {2,4,7}, {4,5,7}

Proof: step (ii)

$N - 1 \geq R_C(M; k - 1)$

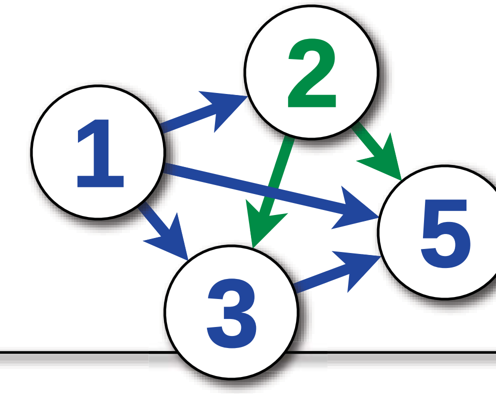
$M \geq G_C(n - 1; k)$

- A fictional example: $N = 7$, $M = 5$, $n = 5$, $k = 3$
  - Original colouring f: {1,2,3}, {1,2,4}, {1,2,5}, {1,2,6}, {1,2,7}, ..., {1,6,7}, {2,3,4}, ..., {5,6,7}
  - Colouring f': {2,3}, {2,4}, {2,5}, {2,6}, {2,7}, ..., {6,7}
  - f'-monochromatic M-subset {2,3,4,5,7} of {2,3,...,N}: {2,3}, {2,4}, {2,5}, {2,7}, ..., {5,7}
  - f-good $(n-1)$-subset {2,4,5,7}: {2,4,5}, {2,4,7}, {4,5,7}
  - {***1***,2,4,5,7} is f-good: {***1***,2,4}, {***1***,2,5}, {***1***,2,7}, ..., {***1***,5,7}, {2,4,5}, {2,4,7}, {4,5,7}

Proof: step (ii) — summary

- **Lemma**: if $R_c(n; k-1)$ is finite for all n then $G_c(n; k)$ is finite for all n
- **Proof**:
  - Induction on n
  - $G_c(k; k)$ is finite
  - We have shown that if $G_c(n-1; k)$ is finite then $G_c(n; k)$ is finite
    - Trick: show that $G_c(n; k) \leq 1 + R_c(G_c(n-1; k); k-1)$

Proof: step (iii)



- **Lemma**: $R_c(n; k) \leq G_c(c(n-1)+1; k)$ for all n, k
- **Proof**:
  - If $N = G_c(c(n-1)+1; k)$, we can find a good subset X with $c(n-1)+1$ elements
  - If k-subset A of X has colour i, put min(A) into slot i
  - E.g.: **{1,2}**, **{1,3}**, **{1,5}**, **{2,3}**, **{2,5}**, **{3,5}**: put 1 and 3 to slot blue, 2 to slot green, 5 to any slot
  - Each slot is monochromatic and at least one slot contains n elements (pigeonhole)!

Ramsey's theorem: proof summary

- $R_c(n; k)$ = smallest N s.t. ∃ monochromatic n-subset
- $G_c(n; k)$ = smallest N s.t. ∃ good n-subset
- **Theorem**: $R_c(n; k)$ is finite for all c, n, k

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then $G_c(n; k)$ is finite for all n

- Induction: $G_c(n; k) \leq 1 + R_c(G_c(n - 1; k); k - 1)$

(iii) $R_c(n; k) \leq G_c(c(n - 1) + 1; k)$ for all n, k

c is fixed